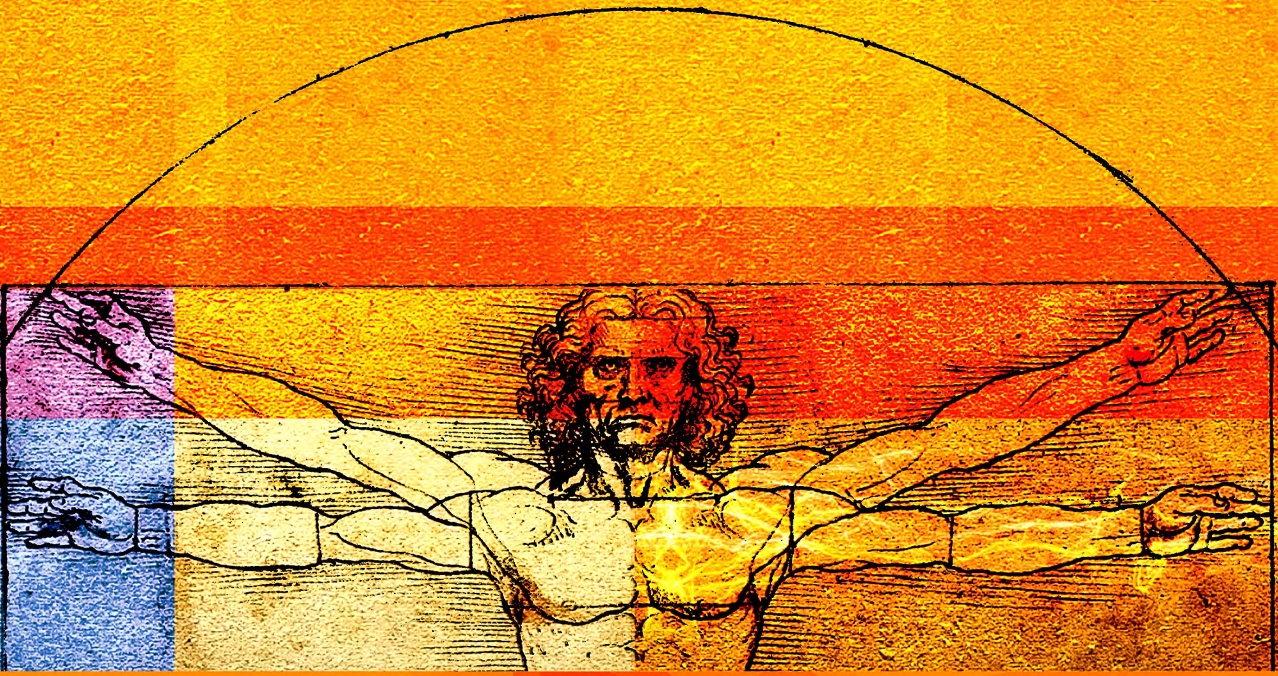

Mathematical Transgressions 2015

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During the last decades of the 20th century, we have entered the Digital Era. The Third Technological Revolution has an enormous socio-economic impact. It affects modern science, including mathematics. It also determines the reforms of national education systems. Mathematicians seek to focus on the so-called concrete mathematics. They explore finite and discrete structures, rather than infinite and continuous ones. They prefer to develop combinatorics and algorithmic thinking, rather than contribute to Bourbaki's edifice. Mathematics itself is expanding its boundaries by merging with computer science, while symbolic computations as well as computer-assisted and automated proofs are transforming it into a quasi-empirical science.

We should agree that mathematics is no longer *the Queen of the Sciences*; while it is still believed to be the basis of modern education, its role needs to be re-defined. It is necessary to address this challenge. The first step in this direction consists of the adoption of a new perspective. The Latin word *transgressio* means an act that goes beyond generally accepted boundaries. This monograph draws together papers written by mathematicians, educators, pedagogues, psychologists, and philosophers. Their aim is to identify a new role of mathematics and mathematics education in the modern world.

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Philosophy and Psychology of Mathematics

Paul Ernest

Challenging three myths about mathematics: Recognising the social responsibility of mathematics

Abstract. In this paper I question and challenge three ideas about mathematics: that mathematics is (1) a unique and unified subject, (2) absolute and value- and ethics-free, (3) an unqualified force for good. Instead I show (1) the motley of meanings pertaining to the name mathematics, (2) how values may be seen to permeate mathematics, and (3) the harm that mathematics can inadvertently cause unless it is applied and taught carefully. Alongside this I acknowledge how mathematics is a widespread force for good. The final recommendation is for the inclusion of the philosophy and ethics of mathematics alongside its teaching at all stages from school to university.

Mathematics is a very rich and powerful subject, with broad and varied footprints across education, science, culture and indeed all of human history. It has excited philosophers and other thinkers since the time of Plato and Euclid or earlier, and remains the subject of much debate. Philosophical discussions and controversies about the nature of mathematics, including mathematical knowledge, truth and the objects of mathematics continue to this day (Hersh, 2007; Kitcher and Aspray, 1988; Tymoczko, 1986). Both academia and society in the large accord mathematics a very high status as an art and as the queen of the sciences (Bell, 1952). Mathematics has a uniquely privileged status in education as the only subject that is taught universally and to all ages in schools. Despite all this exposure and attention it is all too rarely that ideas about the nature of mathematics, how it impacts on society, and its overall role and value in education are examined critically. It is therefore not surprising that there are some widespread myths and misunderstandings

Key words and phrases: philosophy of mathematics, ethics, platonism, social constructivism, absolutism, mathematical harm.

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about mathematics and these roles. My aim here is to uncover and challenge some of the widespread assumptions, ideological presuppositions and myths about mathematics, its role in society, and its impact in the teaching and learning of mathematics. In this paper I question and challenge three ideas about mathematics: that mathematics is

1. a unique and unified subject,
2. absolute, universal and value- and ethics-free,
3. an unqualified force for good.

I claim instead that there is no such thing as mathematics, as the term 'mathematics' does not refer to a single object. Second, I challenge the view that mathematics is absolute, universal and value-free. I argue instead that there are powerful and legitimate reasons for viewing mathematical knowledge, proofs and the objects of mathematics as human constructs. From this perspective, it cannot be claimed that mathematics is ethics and values free. Mathematics education is of course value-laden as are all educational and social activities, they are intended to enhance human life and flourishing. However, my claim is stronger. I will identify some of the ethical values implicit in mathematics itself.

Third, I argue that mathematics does harm as well as good. My claim is that mathematics in school has unintended outcomes in leaving some students feeling inhibited, belittled or rejected by mathematics. In sorting and labelling learners and citizens in modern society, mathematics reduces the life chances of those labelled as mathematical failures or rejects. In addition, even for those successful in mathematics, in shaping thought in an amoral or ethics-free way, mathematics supports instrumentalism and ethics-free governance. This is manifested in warfare, psychopathic corporations, human and environmental exploitation, and in all acts that treats persons as objects rather than moral beings that deserve respect and dignity in all interactions. I conclude that to overcome such myths we need to teach the philosophy and especially the ethics of mathematics alongside mathematics itself.

There is no such thing as mathematics: The myth of mathematics as a unique and unified subject

Ever since Plato, or before, humans have been inclined to view nouns as denoting some real or ideal object. But this is a linguistic and philosophical fallacy. Apart from concrete particulars, far from naming preexistent objects, nouns create fictive objects that correspond to their names, such as Harry Potter, the Equator, or Mathematics. So what can the term mathematics denote? Epistemologically it denotes a body of knowledge. But there is no organised and unified body of mathematical knowledge. Within academic mathematics just pure mathematics alone refers to hundreds of subfields and topics (Davis and Hersh, 1980; Lerman, 2010) that at best share a family resemblance (Wittgenstein, 1953). There is no central set of shared symbols, concepts, theories, proof methods or rhetorical styles (Knuth, 1985). However, even this disparate collection far from exhausts mathematics. There is pure mathematics, applied mathematics, statistics, computer mathematics, school mathematics, accountancy mathematics, street mathematics, ethno-mathematics, and so on. We cannot even say that these all share the same basic number ideas, for apart from all the different systems and structures we label as number (itself another fiction of unified referent) there are knowledges that are constituted by patterning, geometry, argumentation, that can be termed mathematical but do not involve numbers.

Culturally, in terms of social practices, we cannot claim that mathematics constitutes a unity. In addition to all the above disparate knowledge fields, it cannot be said that mathematics originates in a single or unified set of mathematical practices. According to Bishop (1988) all of mathematics evolves from six basic areas of human activity which occur in all societies. These are counting, locating, measuring, designing, playing, and explaining. These practices do not even share a family resemblance and since what we call mathematics originates in these activities, and no subsequent total unification takes place, mathematics remains a multiplicity, a plurality, a motley of disparate knowledge and practices (Wittgenstein, 1956; Lerman, 2010).

These facts notwithstanding, there is a traditional absolutist perspective of mathematics as a unity, a single entity, that can be represented as a skyscraper, with successive storeys of ever more complex theories based on those beneath built on the unshakeable foundations of axioms and logic. Epistemologically, this fails to represent mathematics for technical reasons (Ernest, 1991; 1998). But socially it fails as well, because mathematics has many locations and contexts.

An alternative and better representation of mathematics is a dynamic, growing modern city. This has:

1. Multiple skyscrapers, comprising the different formal theories in mathematics;
2. Universities and academic centers, representing different branches of research mathematics;
3. A business district, representing various applications of mathematics and accountancy and business mathematics;

4. Computers and the city wide internet – computer based mathematics;
5. Favelas and street markets, representing informal mathematical practices on the streets, in the shops and ethnomathematics (D'Ambrosio, 1985);
6. Schools, representing school mathematics and mathematics education.

The city model works both epistemologically, representing the different type of knowledges and associated practices, and socially, indicating the varied social locations and practices that make up mathematics.

The city model illustrates the diversity that goes to make up mathematics, illustrating the fact that the term 'mathematics' does not refer to a single object. This has implications when, for example, we research teachers', students' or the public's beliefs and ideas about the nature of mathematics. It is evident from such people's answers that they do not have the same 'mathematics' in mind. It can be school mathematics, mathematics as used in everyday life, university mathematics, or even research mathematics. Some mathematically literate persons claim to see mathematics in the natural world, the spirals of a sunflower or a nautilus shell, the arc of a rainbow, the juxtaposition of two imaginary dinosaurs with two more making four – even prior to the development of humans able to see it. Before we even consider the disparities between competing philosophies of mathematics we need to acknowledge that there is no such thing as mathematics, and like the plurality in the title for 'mathematics' what we have is a plurality of bodies of knowledge and human practices which we incorrectly form into a fictitious unity.

One of the outcomes of the perspective of mathematics as a plurality of bodies of knowledge originating in and embedded in a wide variety of human practices is that like any other human product mathematics is value and ethics-laden.

The myth of mathematics as absolute, universal and ethics and value-free

It is a matter of controversy as to whether mathematical knowledge is absolutely true, and is made up of truths that are universal and objective. In contrast a growing number of scholars argue that mathematics and mathematical knowledge are human creations (Tymoczko, 1987; Kitcher, 1988; Davis and Hersh, 1980; Ernest, 1998). Tracing the evolution of mathematical concepts and mathematical truth-methods through history we can demonstrate that there is no unchanging body of knowledge and truth (Wilder, 1974). As well as the historical addition of new concepts, results and proof methods to the sum of mathematical knowledge, which all philosophies of mathematics accept, there are also deep seated changes to the concepts, meanings, results and proof methods of mathematics including logic and what counts as acceptable mathematical proofs (Gillies, 1992; Lakatos, 1976). From this humanistic perspective, although theorems and warranted mathematical truths are humanly certain, just as mate in two moves can be from a particular chess position, mathematical truth cannot claim to be absolute, eternal, objective universal and superhuman (Ernest, 1998; Hersh, 1997). Such a human-centric view of mathematics implies, like all other human artefacts constructions and practices, that mathematics is value-laden and an ethical enterprise.

Elsewhere I have identified a range of different values embedded in pure mathematical knowledge, including epistemological or epistemic, ontological or ontic, aesthetic and ethical values (Ernest, 2015). Here I wish only to argue the case for ethical values, namely that mathematical knowledge is embedded with implicit ethical values.

There are two sorts of ethical values that I wish to identify within mathematics. First there are ethical values that are independent of any particular philosophical orientation that I wish to identify in mathematics. I term these general mathematical ethics. Second, there are those values that follow on from a social constructivist philosophy of mathematics, including humanistic, fallibilist or anti-absolutist philosophies of mathematics. For such philosophies the job is half done, for if mathematical knowledge is humanly constructed, then it is not a far step to argue, as I do here, that like all other human constructions mathematics is imbued with human and hence ethical values. I term these social constructivist ethics.

A. General mathematical ethics

I wish to claim that ethical values imbue both pure and applied mathematics. At the heart of mathematics lies mathematical knowledge, that is, justified mathematical propositions and their proofs. Proof and justification are arguments and reasoning applied to persuade, indeed to convince other persons about the truth of mathematical claims, that is, that mathematical theorems are adequately warranted. However, the very use of proof and justification can be said to embody the values of openness, fairness and democracy. Proof itself embodies democracy because it opens up the basis for knowledge to all for scrutiny and verification. Whether it is the shop keeper presenting a calculated bill for purchases or a mathematician publishing her latest theorem, the written account allows scrutiny of the correctness of the claims and reasoning. Indeed the terms justification and justice have the same roots in English. From the 14th century on justification has meant the action of justifying and the administration of justice, and justice is the quality of being fair and just – the exercise of authority in vindication of what is right (Harper, n.d.). True justice depends on the open justification of decisions which is the basis of both mathematics and democracy. Mathematics, like democracy, is fair because of this openness and potentially equal treatment of all with respect to knowledge claims, their warranting and decisions as to their status as knowledge. This is not an incidental outcome of proof, but implicit in the very nature and purpose of proof, that is to convince others.

There is an analogy with the physical sciences. For acceptability, scientific claims should be based on observations or experiments that are replicable. That is different scientists should be able to reach the same conclusions independently. Likewise, mathematical claims should be based on proofs which convince different mathematicians of their validity. Thus at the heart of both mathematics and science lie the values of openness, fairness and democracy. However, in mathematics, no special apparatus is needed to validate mathematical knowledge claims, just a well informed mathematician.

Mathematics has long been associated with ideas of justice and fairness. In ancient societies including those in Mesopotamia the reliability of calculation, measures and numerical records was understood as part of the idea of justice (Høyrup, 1994). Later on, in ancient Greece, mathematical proof emerged out of a background of philosophical argument and reason that developed with the first, albeit limited, democracy with its justification of human claims and rights. It has been argued that some mathematical concepts and methods embody ideas of fairness. Johnson (2012) argues that fairness underpins probabilistic concepts and probabilistic methods of reasoning, and that this has implications for the history and present day practices of market trading.

My claim is that by its very nature, mathematics embodies, displays and transmits the values of openness, fairness and democracy, admittedly in a restricted form as I have explained above. But even the identification of the slenderest strand of ethics and values within the body of mathematical knowledge, and my claim is not so modest, proves my point that mathematics is not value and ethics-free.

I also wish to argue that even mathematical research conducted purely for its own sake is ethical. Pure mathematical research is conducted simply to expand human knowledge and to satisfy the professional drive, expertise and mastery of the intrinsically motivated mathematician. But expanding human knowledge is for the good of humankind, and expanding the skills and mastery of the mathematician is also for the good. Striving for human flourishing and for the good is the central goal of ethics. Since the development of pure mathematics entails this goal, it is a fundamentally ethical enterprise.

B. Social constructivist ethics

Before addressing the issue of ethics, it is important to establish what the nature of mathematics is claimed to be from a social constructivist perspective. In brief, the social constructivist philosophy of mathematics contends that mathematics, including mathematical knowledge, concepts and rules are humanly constructed and invented tools (Ernest 1998, Hersh 1997). Mathematics is based on human practices, which is why it is so highly applicable, but developed and presented in a highly objectivised form. Social constructivism contends that pure mathematics is a distillate of the applied mathematical practices that preceded it historically, although once established pure mathematical practices take on a life of their own. Mathematicians extend and develop mathematical knowledge and methods for their own sake, which I have argued, is an ethical undertaking. Through being socially constructed in a variety of practices mathematics is of necessity linked with other knowledge areas, consequently in addition to any values already present in mathematical knowledge, the values saturating adjacent knowledge areas and practices seep into mathematics, rendering it richly value-laden. The products of any ongoing human practices are value-laden and ethical, and mathematics is not hermetically sealed and no more exempt from this than any other human products.

Given that mathematics is not purely abstract knowledge existing in some objective realm of existence, the questions arise. What can it be? Of what 'stuff' is it made? What is the ontology of mathematics if it does not consist of abstract propositions and concepts in some superhuman Platonic realm? According to social constructivism mathematics and mathematical knowledge is made up of the coordinated combination of three material things. These are:

1. Sign systems, existing through their markings (their tokens) but understood as abstract types through their relationships in/with the following two components,
2. Individual personal meanings, that is the sense persons have learned to make of the signs and sign-systems, and their creative usage, including the understanding which signs (within their contexts) are to be regarded as equivalent, and

3. The social institutions comprising the rules and meanings concerning the uses of the sign-systems of mathematics (including which signs may be conjoined or otherwise lawfully related to other signs by such relations as name elaboration, simplification with equivalence, calculation, deduction, etc.).

These three coordinated elements set the scene for the underlying unit for social constructivism, namely conversation, in which texts made up of signs are exchanged between persons in social contexts.

One of the central types of texts utilised in mathematics is the mathematical proof. The role of proof is epistemological in establishing the truth of a theorem. From the perspective of humanism (Hersh, 1993), social constructivism (Ernest, 1998), or compatible philosophies of mathematics, convincing persons about the correctness of reasoning in a proof lies at the heart of the epistemological function of proof. A demonstration of correctness of reasoning is always addressed to another. Consequently it is not surprising that in the development of a social constructivist philosophy of mathematics elsewhere (Ernest, 1991; 1998). I propose conversation as the underlying epistemological unit. The social constructivist account of the conversational basis of mathematics draws on the work of Wittgenstein (1953) and Lakatos' (1976) *Logic of Mathematical Discovery*. My claim is that conversation, consisting of symbolically mediated exchanges between persons, underpins mathematics, and that it does so in four distinct ways.

1. The ancient origins and various modern systems of proof are conversational, through dialectic or dialogical reasoning, involving the persuasion of others.
2. Mathematics is primarily a symbolic activity, using written inscription and language and inevitably addressing a reader, so mathematical knowledge representations are conversational.
3. A substantial class of mathematical concepts have a conversational structure (e.g., epsilon-delta definitions of limit in analysis, hypothesis testing in statistics, as well as other concepts, Ernest 1994a).
4. The epistemological and methodological foundations and acceptance of mathematical knowledge, including the nature and mechanisms of mathematical knowledge genesis and warranting are accounted for by social constructivism through the deployment of conversation in an explicitly and constitutively dialectical way.

My argument is that the very content of mathematical knowledge – its concepts, methods, proofs – are conversational, so conversation cannot be dismissed as merely part of the context of discovery (Popper, 1959). These contents as well as the conversational warranting mechanisms described in

Lakatos' (1976) *Logic of Mathematical Discovery* and in Ernest's (1998) *Generalised Logic of Mathematical Discovery* are also part of the context of justification (Popper, 1959). So mathematics in all of its manifestations is riven through and through by conversation, throughout its origins, practices, and throughout abstracted mathematical knowledge itself.

The published version of a proof appears monological because all of the anticipated criticisms and responses have been overcome and incorporated in the final polished result. But as Lakatos (1976) shows the hidden dialogic of the proof leaves its mark in the refined definitions and lemmas that make up the final proof. It might be argued that as conversation is subsumed into mathematics it becomes vestigial and its ethical dimensions become attenuated and discountable. My rejoinder is threefold. First of all, although conversation originates in its observable interpersonal form it becomes internalized as the structure of our thinking, so that all of our thought is shaped conversationally through this continuing process, with all of its concomitant assumptions and connotations. Second, conversation does not become vestigial because of its continuing roles in the warranting of mathematical knowledge. Furthermore, the warranting of mathematical knowledge never ceases, as every new formulation or publication in mathematics requires warranting. Third, as I have argued above, from humanistic and social constructivist perspectives the distinction between the contexts of discovery and justification can no longer be claimed to be watertight or absolute. Some values from the context of discovery cannot be prevented from imbuing the context of justification. Thus mathematical knowledge and the processes and products of the context of justification are laden with the values of conversation, and more generally with human values, as argued above.

As I have argued, in a number of ways, conversation lies at the heart of mathematics, providing it with a human foundation. It is intrinsic to the fabric of mathematics, underpinning its concepts and objects, representations, genesis, proof and warranting. But conversation as an interpersonal activity is inescapably ethical, it is not just about exchanging information (Ernest, 1994b; Johannesen, 1996; Gadamer, 1986; Rorty, 1979). For it entails engaging with a speaker or listener as another human being with mutual respect and trust, attending to another's proposals and responding relevantly, and being aware of reactions to one's own contributions. In mathematics, putting one's proposals in an appropriate and accessible format following received norms of acceptability is part of one's ethical responsibility throughout pure, applied and educational mathematics.

Overall, my claim is that in a number of ways mathematics is imbued with ethical values. Its basis in verifiable truth claims means that it is shot through with the values of openness and democracy. This holds no matter what philosophy of mathematics is adopted. In addition, from the social constructivist perspective, the nature of mathematics as a symbolic activity, a specialized and supplemented form of written language means that the ethics of human communications are presupposed. If mathematics is conversational then like all forms of inter-human activities and relationships, it is inescapably ethical.

Is mathematics an untramelled good?

The third myth I wish to challenge is that mathematics is an untramelled good, and that promoting mathematics leads solely to beneficial outcomes. The received wisdom dominating the institutions of mathematics, mathematics education and society in general is that mathematics of itself is a wonderful boon for all of humankind, and in areas where its positive benefits are not remarked it is simply neutral (Gowers, n. d.). Instead I wish to ask what are or might be the actual outcomes and potential costs of elevating and privileging mathematics in education and society, including any unintended outcomes? Looking at such outcomes, does mathematics cause any harm or evil? To mathematicians and many others even asking this question, let alone answering it in the affirmative, might seem unthinkable, a ridiculous questioning of what has hitherto been unquestionable. To educationists it is not so difficult ask this question, or even to answer it in the affirmative, when the impact on disadvantaged students and society is considered (Stanic, 1989).

Before I address the potential harm that mathematics may do, let me begin by affirming that mathematics has great value. The overall value of mathematics comprises the benefits and goods it offers to humanity as a whole. There are two types of value of mathematics. First, there is the intrinsic value that mathematics has as a discipline or area of knowledge, the value of mathematics purely for its own sake. Second, there is extrinsic value, the general social value of mathematics on the basis of its applications and uses in society. In addition to the social benefits of its applications mathematics also has personal value. This is the value of mathematics for learners and for other persons more widely as it plays out in terms of individual benefit. Such benefits will vary across individuals according to personal circumstances, experiences and so on.

The intrinsic value of mathematics

Mathematics has intrinsic value, and as I argued above the furthering of mathematics for its own sake is an ethical good for humankind. Mathematics is a powerful exploration of pure thought, truth and ideas for their intrinsic beauty, intellectual power and interest. In its development mathematics creates and describes a wondrous world of beautiful crystalline forms that stretch off to infinity in richly etched exquisiteness. Part of the intrinsic value of pure mathematics is its widely appreciated beauty (Ernest, 2015). "Like painting and poetry mathematics has permanent aesthetic value" (Hardy, 1941, p. 14). "Mathematics possesses not only truth, but supreme beauty – a beauty cold and austere, like that of sculpture" (Russell, 1919, p. 60).

These virtues and values are appreciated not only by those initiated into the most exclusive inner sanctum of mathematics, the area occupied by the ground-breaking creative mathematicians. We are often confronted with complex and fascinating mathematics-based images in the media, for example multi-coloured pictures of fractals, complex tessellations and other beautiful representations. These contribute to the public perception that mathematics can be both beautiful and intriguing, and has an intrinsic value.

The extrinsic and social value of mathematics

It is universally acknowledged that mathematics provides the foundation for much of knowledge, especially science, engineering, and information and communication technologies. The essential role of mathematics throughout society is demonstrated by a consideration of three domains of application: science, computing and finance, although more could be cited.

First, with regards to science, mathematics is known as both the queen and servant of science (Bell, 1952). As its servant mathematics provides the language by means of which modern science is formulated. Models, laws, theories and predications going as far back as 2000 years ago to the Ptolemaic model of the universe could not be expressed without mathematics. Furthermore, scientific applications based in mathematics underpin engineering, technologyand the whole material basis for modern life.

Second, enlarging on the theme of technology, computing and the information and communication technologies that form the language and basis for all our modern media, knowledge systems and control mechanisms are wholly based on mathematics. Both the knowledge representations and the programmed instructions upon which information and communication technology depends can only be expressed by means of the coding and logic supplied by mathematics.

Third, and far from least, finance, economics, trade, business and through them, social organisation, rest on a mathematical foundation.

The tangible embodiment of economics, namely money, is the lifeblood that circulates throughout these bodies and activities. The commercial basis of modern society simply would not be possible without money and hence mathematics.

Each of these three domains of application undoubtedly has many great benefits in terms of human flourishing, including improvements in health, nutrition, housing, transport, agriculture, manufacturing, education, leisure, communications and wealth. Undoubtedly more human beings than ever live longer, healthier, better educated, more comfortably and wealthier as a consequence of the mathematics-led developments in the sciences, technology and engineering in the past two centuries.

In addition to these social benefits shared by so many, mathematics has great personal value. Learners and more widely, other persons, benefit from mathematics as: an enlarging element of human culture, a means of personal development and growth, a valuable tool for use in socially, both as workers, and general citizens in society, and a means of gaining certification for entry to employment or further education.

We live in a mathematized social world, and mathematics is the basis for virtually all of modern life. The immense utility of mathematics must be acknowledged as a great strength and virtue. For without it not only would we have to forego many of the tools we as individuals and society rely on, but many of the necessities and much of our prosperity would disappear. Mathematics is arguably the most generally applicable of all human knowledge fields and the good qualities of modern living depend on it.

Features and characteristics of mathematics

An immediate question is what are the components and dimensions of mathematics that contribute to its great intrinsic and extrinsic value? The most obvious is that of number and calculation. Calculation is central to mathematics, in that it dominates history and schooling. Mathematics as a scientific discipline is claimed to originate around 3000 years BCE (Høyrup, 1980). Thus it was already halfway through its history (C. 500 years BCE) before proof entered into mathematics. Prior to that number recording and calculation, including some geometric measurement constituted the totality of mathematics. Even since then, numbers and calculation have dominated both the practical uses of mathematics and its educational content, with Euclidean geometry overall playing a minor role, and that just in elite education

At the heart of calculation are rulebased general procedures in which the meaning of numerals, especially their place-value meaning, by virtue of their relative positioning is ignored. Further, largely as a result of Islamic contributions, algebra emerged in the middle ages providing the abstract language of mathematics upon which all modern developments depend. Algebra is primarily generalized arithmetic in origin and is subject to generalized arithmetical procedures and rules, and its strength is that specific meanings are detached. This was explicitly noted over 300 years ago by Bishop Berkeley.

> ... in Algebra, in which, though a particular quantity be marked by each letter, yet to proceed right it is not requisite that in every step each letter suggest to your thoughts that particular quantity it was appointed to stand for.
>
> (Berkeley, 1710, p. 59)

At its heart, algebra is variable based, thus forcing a unique linguistic move in the language away from specific values and meanings to general rules and procedures. This move has some great benefits. It enables the miracle of electronic computing in which mathematical rules and procedures are wholly automated and no reference to or comprehension of the meaning of mathematical expressions is required.

A further characteristic of school, university and research mathematics is that they are represented in the symbolism and language of mathematics, fundamentally in sentences. Mathematical sentences, although often containing symbols, conform to the usual subject-verb form, or more generally, in terms-relation form, where a relation is a generalised verb. In a detailed analysis Rotman (1993) has found that usually, although there is some limited use of the indicative mood, the predominant verb form in mathematical language is the imperative mood. Imperatives instruct or direct actions – either inclusively, such as: let us ..., consider ... or exclusively, such as: add, count, solve, prove, etc. Mathematics is more richly studded with imperatives than any other school subject (Rotman, 1993; Ernest, 1998). Mathematical operations require rigid rule following. At its most creative mathematics allows choices among multiple strategies, but each of the lines pursued involves strict rule following. Mathematics is very unforgiving too. There is no redundancy in its language and any errors in rule following derails the procedures and processes. The net result is a social training in obedience, an apprenticeship in strict subservience to the printed page. Mathematics is not the only subject that plays this role but it is by far the most important in view of its imperative rich and rule-governed character.

One of the most important ways that a social training in obedience is achieved is through the universal teaching and learning of mathematics from a very early age and throughout the school years. The central and universal role of arithmetic in schooling provides the symbolic tools for quantified thought, including not only the ability to conceptualize situations quantitatively, but a compulsion to do so. This compulsion first comes from without, but is appropriated, internalized and elaborated as part of the postmodern citizen's identity. We cannot stop calculating and assigning quantified values to everything, in a society in which what matters is what *counts* or is *counted*.

The teaching and learning of mathematics in schools, and thus the development of mathematical identity requires that, from the age of five or soon after, depending on the country, children will (Ernest, 2007):

1. acquire an object-oriented language of objects and processes,
2. learn to conduct operations on and with them without any intrinsic reasons or sense of value (deferred meaning),
3. decontextualise their world of experience and replace it by a deliberately unrealistic and very stylized model composed of simplified static objects and reversible processes,
4. suppress subjectivity, experiential being and feelings in their mathematical operations on objects, processes and models,
5. learn to prioritize and value the outcomes of such modelling above any personal or connected values and feelings, and apply these outcomes irrespective of such subjective dimensions to domains including the human 'for your [their] own good' (Miller, 1983).

King (1982) researched the mathematics in 5-6 year old infant classrooms. He found that mathematics involves and legitimates the suspension of conventional reality more than any other school subject. People are coloured in with red and blue faces. "A class exercise on measuring height became a histogram. Marbles, acorns, shells, fingers and other counters become figures on a page, objects become numbers" (King, 1982, p. 244). In the world of school mathematics even the meanings of the simplified representations of reality that emerge are dispensable.

> Most teachers were aware that some children could not read the instructions properly, but suggested they 'know how to do it (the mathematics) without it.' ... Only in mathematics could words be left meaningless.
>
> (King, 1982, p. 244)

In the psychology of mathematics education instrumental understanding, consisting of knowing how to carry out procedures without understanding, versus relational understanding that also comprises knowing how and why such procedures work, is much discussed as a problem issue (Skemp, 1976; Mellin-Olsen, 1987). It is no coincidence that what is termed instrumental understanding is also a form of the instrumental reasoning critiqued by the Frankfurt School, as reported below.

In summary, many procedures on signs are carried out with abstracted or deferred meanings, and many mathematical texts, be they calculations, derivations or proofs, involve the reader following rule-governed sequences or orders. In education mathematics is the subject most divorced from everyday or experienced meaning, and the objectification and dehumanisation of the subject are a necessary part of its acquisition.

However, I need to qualify these claims. Although mathematical signs and procedures are detached from meaningful referents in the world, mathematics creates its own inner world of meanings. Mathematicians work within a richly populated conceptual universe which is very meaningful for them. Success at mathematics at most levels is associated with persons involved having a meaningful domain for the interpretation of mathematical signs and symbols, even if it is within the closed world of mathematics. Furthermore, applied mathematicians interpret mathematical models in the world around us so in applications meanings are reattached. Likewise, although mathematical language is very rich in imperatives, successful users of mathematics at all levels have certain degrees of freedom available to them, such as which methods and procedures to apply in solving problems. These qualifications notwithstanding, the study of mathematics does instil both the capacity to, and the expectation of, meaning detachment during reasoning and calculative procedures. Likewise, it does prepare its readers to follow the imperatives in the text during the technical and instrumental reasoning involved in mathematics.

Mathematical thinking as detached instrumental and calculative reasoning

My claim is that the linguistic characteristics and moves indicated above have costs, including unanticipated negative outcomes when extended and applied beyond mathematics. For as I have argued, the mathematical way of thinking promotes a mode of reasoning in which there is a detachment of meaning. Reasoning without meanings provides a training in ethics-free thought. Ethical neutrality or irrelevance is presupposed because meanings, contexts and their associated purposes and values are stripped away and discounted as irrelevant to the task or thought in hand. Furthermore, as I argued above, there is a widespread perception of mathematics as absolute, universal and imbued with certainty, and hence an ethics and value-free domain of thought. This is the second of two myths that I critiqued above. Such perspectives and reasoning contribute to a dehumanized outlook, for without meanings, values or ethical considerations reasoning can become mechanical and technical or thing- or object-orientated. These modes of thinking foster what have been termed separated values.

Gilligan (1982) proposes a theory of values which can usefully be applied to mathematical and other types of reasoning. This theory distinguishes separated from connected values positions and places them in opposition. The separated position valorises rules, abstraction, objectification, impersonality, unfeelingness, dispassionate reason and analysis, and tends to be atomistic and thing-centred in focus. The connected position is based on and valorises relationships, connections, empathy, caring, feelings and intuition, and tends to holistic and human-centred in its concerns. These two values positions can be seen as pairs of oppositions, with separated values (first) contrasted with connected values (second, respectively): rules vs. relationships, abstraction vs. personal connections, objectification vs. empathy, impersonal vs. human, unfeeling vs. caring, atomistic vs. holistic, dispassionate reason vs. feelings, analysis vs. intuition.

The separated values position applies well to mathematics. Mathematical objects are entities resulting from objectification and abstraction and are naturally impersonal and unfeeling. Mathematical structures are constituted by abstract and rule-based sets of objects and their structural relationships. The processes of mathematics are atomistic and object-centred, based on dispassionate analysis and reason in which personal feelings play no direct part. Thus separated values fit mathematics very well and indeed can be said to be an essential part of mathematics. Mathematics both embodies and transmits these values.

Separated values and the associated outlooks are necessary, indeed essential by the very nature of mathematics, and their acquisition constitute assets and are undoubtedly beneficial for thinking in mathematics. A separated scientific outlook is also useful in reasoning in other inanimate domains, such as in physics and chemistry, where atomistic analysis, strictly causal relationships and structural regularities yield high levels of knowledge. However, thinking exclusively in the separated mode can lead to problems and abuses when applied outside mathematics and the physical sciences to society. In the human sphere exclusively separated values are unnecessary and potentially harmful, since they factor out the human and ethical dimensions. In seeing the world mathematically the beautiful richness of nature and human worlds with all their contextual complexity and ethical responsibilities, are replaced by simplified abstracted and objectified structural models. Although mathematical perspectives and models are powerful and useful tools for actions in the world, including the improvement of human life conditions, when overextended they can become a threat to our humanity. Inculcating these values can lead to a dehumanized outlook when applied to social and human worlds. Furthermore, separated values extended too far beyond mathematics imply that mathematics and its applications have no ethical or social responsibility.

The vision I want to develop is that subjection to mathematics in schooling from halfway through one's first decade, to near the end of one's second decade, and beyond if one so chooses, structures and transforms our modes of thought in ways that may not be wholly beneficial. I do not claim that mathematics itself is harmful. But the manner in which the mathematical way of seeing things and relating to the world of our experience is integrated into schooling, society and above all the interpersonal and power relations in society results in the transformation of the human outlook. This is a contingency, an historical construction, that results from the way that mathematics has been recruited into systems thinking instead of empathising (Baron-Cohen, 2003) and separated values instead of connected values (Gilligan, 1982) that dominate western bureaucratic thinking. It also results from the way mathematics serves a culture of objectification, termed a culture of *having* rather than *being* by the critical theorist Fromm (1978).

One framework that subsumes these aspects of the application of mathematics is that of instrumental reason or rationality. Instrumental reason is the objective form of action or thought which treats its objects simply as a means and not as an end in itself. It focuses on the most efficient or most cost-effective means to achieve a specific end, without reflecting on the value of that end (Blunden, n. d.). Instrumental reason has been subjected to critique by a range of philosophers from Weber to Habermas (Schecter, 2010). This includes Heidegger, who argues that instrumental reason and what he terms *calculative thinking* lead us into enclosed systems of thought with no room for considering the ends, values and indeed ethical dimensions of our actions (Haynes, 2008). As Heidegger puts it, even 'the world now appears as an object open to the attacks of calculative thought' (Dreyfus, 2004, p. 54).

A broader critique comes from the Critical Theorists of the Frankfurt School (including Adorno, Fromm, Habermas, Horkheimer and Marcuse) who see instrumental reason as the dominant form of thought within modern society (Bohman, 2005; Corradetti, n. d.). By focussing on technical means and not on the ends of their actions, persons, governments and corporations risk complicity in the treatment of human beings as objects to be manipulated, in actions that threaten social well-being, the environment and nature. It underpins behaviours of some governments and multinational corporations in reducing costs and chasing profits without regards for the human costs. Such actions by corporations have been termed psychopathic (Bakan, 2004). We are now so used to the economic, instrumental model of life and human governance that most persons see it as an unquestionable practical reality, a necessary evil and are not shocked or outraged.

Much of the Frankfurt School critique was prompted by the rise of Nazism in Germany, with its authoritarian leaders (Adorno *et al*, 1950) and the heartless complicity of ordinary citizens in Germany and occupied territories before and during World War 2. The capture, transportation, enslavement and murder of millions of fellow citizens was not simply undertaken by monsters. These wholesale activities would not have been possible without many ordinary citizens unquestioningly doing their everyday jobs as part of this monstrous programme. Arendt (1963) terms this ordinariness, from the actions of Eichmann downward, the 'banality of evil'. The fact that many ordinary citizens were highly educated did not prevent them from complicity in mass murder. As Dr. Haim Ginott, a school principal who survived a Nazi concentration camp, wrote in his advice to his teachers:

> I am a survivor of a concentration camp. My eyes saw what no man should witness: gas chambers built by learned, children poisoned by educated physicians, infants killed by trained nurses engineers, women and babies shot and burned by high school and college graduates. So I am suspicious of education. My request is: help your students to become human. Your efforts must never produce learned monsters, skilled psychopaths, educated Eichmanns. Reading, writing and arithmetic are important only if they serve to make our children more humane.
>
> (Ginott, 1972, page unknown)

My argument is that mathematics plays a central role in normalizing instrumental and calculative ways of seeing and thinking. From the very start of their education children are schooled in these ways of seeing and being. As I have argued, the detachment of meaning and the following of imperatives in mathematical texts provides the central platform for instrumental thought. [1]

There is a further factor too. Among philosophers, mathematicians, as well as in school and society more generally, mathematics has the image of objectivity, unquestionable certainty, with claims being settled decisively as either true or false as well as being ethically neutral (Ernest, 1998; Hersh, 1997). This is what I identified and critiqued above as the sec-

1. Of course the right social circumstances are needed too. A society with values of strong social-conformity and a culture of obedience to authority is needed, as Milgram (1974) showed in his experiments. However, as I have argued, subjection to thousands of hours of school mathematics and schooling in general will contribute to this.

ond mathematical myth. Thus a training in mathematics is also a training in accepting that complex problems can be solved unambiguously with clear-cut right or wrong answers, with solution methods that lead to unique correct solutions. Within the domain of mathematical reasoning, problems, methods and solutions are value-free and ethically neutral. But carrying these beliefs beyond mathematics to the more complex and ambiguous problems of the human world leads to a false sense of certainty, and encourages an instrumental and technical approach to daily problems. This is damaging, for when decision making is driven purely by a separated, instrumental rationality, then ethics, caring and human values are neglected, if not left out of the picture altogether. Kelman (1973) argues that ethical considerations are eroded when three conditions are present: namely, standardization, routinization, and dehumanization. Since mathematics is the essence of instrumental reason, with its focus on means to ends and not on underlying values, and its procedures require standardization, routinization, and dehumanization, the concomitant erasure of ethics is no surprise. Thus a training in mathematical thinking, when mis-applied beyond its domain of validity to the social domain, is potentially damaging and harmful.

The social impacts of mathematics and its application

One of the key areas in which instrumental modes of thinking is widespread is within the applications of mathematics. I have described some of the broad range of applications of mathematics in society and their widespread benefits. Alongside these beneficial outcomes the qualifications and caveats I offer here are relatively small, but nevertheless significant negative outcomes. The direct applications of mathematics underpin science, technology including information and communication technologies, and finance and business. Thus, for example, mathematics underpins military applications such as nuclear weapons, missile guidance systems, battlefield computer systems, drone technologies, and so on. I am not claiming bad uses of such weapons makes mathematics and science evil. This would be fallacious. But I am claiming that applied mathematicians should try to be aware of the uses to which their applications are made, and if they are potentially hurtful or harmful should at least consider the consequences and their own involvement as facilitators. It has been suggested that there should be a Hippocratic oath for mathematicians (Davis, 1988). Given the widespread views of the neutrality of mathematics, even of applied mathematics, this would seem to be an unlikely development.

There is an outstanding use of mathematics that is not usually counted among the applications of mathematics. This is the role of mathematics as basis of money and finance. Money and thus mathematics is the tool for the distribution of wealth. It can therefore be argued that as the key underpinning tool mathematics is implicated in the global disparities in wealth and life chances manifested in the human world. It is not an exaggeration to claim that many current forms of capitalism distort equality in and across global societies to the detriment of social justice, as well as promoting consumerism. Of course this is a hot political issue. My argument is not that we should oppose the western capitalist system like the Anti-Globalization and Occupy movements (Wikipedia, n. d. a, b). Instead my proposal is that we should foster an ethical and in particular a critical, social justice oriented attitude towards applications alongside mathematical skills so that students and citizens in our democracies can make up their own minds. There is a substantial literature on critical mathematics education that promotes this goal (Ernest, 1991; Skovsmose, 1994; Powell and Frankenstein, 1997; Ernest et al., 2016). Furthermore, the idea that our actions should be ethical and in particular promote social justice is now mainstream thinking, at least in Europe, for example the European Union Treaty stipulates that it shall promote social justice (European Union, n. d.).

The social impact of the image of mathematics

An indirect way through which mathematics impacts on society and individuals is through its images, which can be divided into social and personal images. Social images of mathematics include public images, which are representations in the mass media, such as film, cartoon, pictorial, and computer representations of mathematics. They also include school images which incorporate classroom posters, equipment, textbook teacher presentations, and school mathematical activities as experienced by the learners. Parent, peer or others' narratives about mathematics also contribute to its social image. Personal images of mathematics include mental pictures, visual, verbal or other mental representations, and are assumed to originate from past experiences and encounters with mathematics, as well as from social talk and other public representations. Personal images of mathematics comprise both cognitive and affective dimensions and effects. The types of mathematics as portrayed in its images can include research mathematics and mathematicians, school mathematics, and mathematical applications, both everyday or more complex. Social and personal images of mathematics are intimately related, as personal images must be assumed to result from the lived experiences of learning and using mathematics and from exposure to social images of mathematics. Likewise, social images of mathematics are constructed by individuals or groups drawing on their own personal images, which are then represented and made public. Both kinds of image can have implicit elements of which individuals are not explicitly aware. Thus, what is termed the hidden curriculum comprises those accidental or unplanned elements of knowledge representations and learning experiences within the school curriculum, which can include images of mathematics (Ernest, 2012).

A widespread public image of mathematics in the West is that it is difficult, cold, abstract, theoretical, ultra-rational, but important and largely masculine (Buerk, 1982; Buxton, 1981; Ernest, 1996; Picker and Berry, 2000)). It also has the image of being remote and inaccessible to all but a few super-intelligent beings with 'mathematical minds'. For many people the image of mathematics is also associated with anxiety and failure. When Brigid Sewell was gathering data on adult numeracy for the Cockcroft Inquiry (1982) she asked a sample of adults on the street if they would answer some questions. Half of them refused to answer further questions when they understood it was about mathematics, suggesting negative attitudes. Extremely negative attitudes such as 'mathephobia' (Maxwell, 1989) probably only occur in a small minority in western societies, but are nevertheless a significant extreme within the distribution of attitudes. Thus some of the problems associated with widespread social (and personal) images of mathematics are the perceptions that it is a masculine subject, much more accessible to males; and that it is a difficult subject only accessible to a gifted minority. The effect of these images, coupled with the negative learning experiences reported by some students, is to foster negative personal images of mathematics incorporating negative attitudes such as poor confidence, lack of self-efficacy beliefs, and dislike and even anxiety with respect to mathematics. One of the contributors to the negative images of mathematics is the absolutist image of mathematics as objective, superhuman and value-free, critiqued as the second myth above. For many this contributes to a sense of alienation and exclusion from mathematics (Buerk, 1982; Buxton, 1981). Of course, for a successful minority this image is part of the attraction of mathematics, namely that it is unchanging, perfect, and a safe haven from the chaos and uncertainties of everyday life.

Two of the detrimental effects of images of mathematics are thus the masculine image of mathematics with its negative impact on female students, and the negative impact of mathematics on the attitudes and self-esteem of a minority. The problem with these negative impacts is that mathematics is a highly esteemed and valued subject in schools and universities. Because of this, mathematics examinations are used as a sifting or filtration device in society, and life chances and social rewards are disproportionately correlated with success at mathematics. Sells (1973, 1978) has termed mathematics the 'critical filter' in determining life-chances. While mathematical knowledge has important uses and applications in modern societies, the status of mathematical achievement is elevated beyond its actual utility. Mathematics is increasingly hidden from citizens in modern society behind complex systems including information and communication technology applications, and the immense computerised control and surveillance systems that regulate and monitor modern societies. Advanced mathematical skills are not needed by the many that operate these systems, and can do so successfully without awareness of their mathematical foundations (Niss, 1994; Skovsmose, 1988).

In addition, success in school mathematics is strongly correlated with the socio-economic status or social class background of students. Although this is true with virtually all academic school subjects, mathematics has a privileged status. It is the examinations in mathematics in particular that serve as a fractional distillation device that is class reproductive, at least to some extent. Talented mathematicians from any background may be successful in life, nevertheless the net effect of mathematical examinations is the grading of students into a hierarchy with respect to life chances. This hierarchy doubly correlates with social class socio-economic status and social class in terms of both the social origins and the social destinations of students So it is not merely raw mathematical talent that is reflected in mathematical achievement. It is also partially mediated by cultural capital (Bourdieu, 1986; Zevenbergen, 1998). My claim is that the social image of mathematics as experienced by learners contributes to their personal image of mathematics and that this is an important factor in their success in mathematics. Personal images of mathematics include attitudes to mathematics and attitudes to mathematics play a key role in success at mathematics via multiplying mechanisms which I call the success and failure cycles (Ernest, 2013).

The mechanisms are as follows. Some students suffer from negative attitudes to mathematics, including poor confidence and poor mathematical self-concept, and in a minority possible mathematics anxiety (Buxton, 1981). Following Maslow's (1954) hierarchy of needs theory, persons will do a great deal to avoid risks including threats to personal self-esteem. So negative attitudes lead to reduced persistence and some degree of mathematics avoidance resulting in reduced learning opportunities. A consequence of this is lack of success in mathematics including failure. Students who experience an overall lack of success and repeated failure at mathematical tasks and tests develop or strengthen their negative attitudes to mathematics, completing a self-reinforcing cycle, leading to a downward spiral in all three of its components, illustrated in Fig. 1.

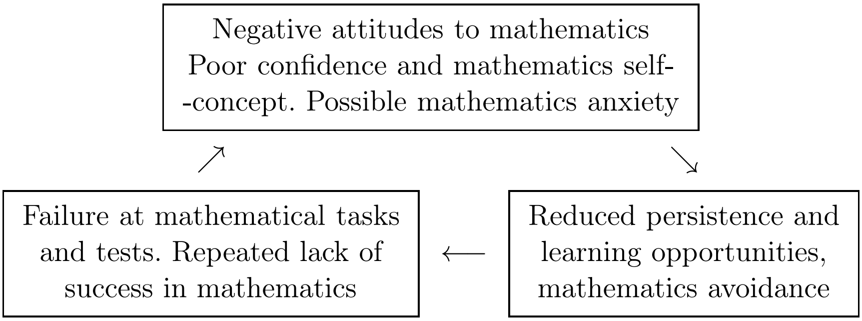

Figure 1. The Failure Cycle (adapted from Ernest 2013).

In this, as in any proper cycle, there is no beginning point. All three elements develop together, and any one of them could be nominated as a starting point. Failure leads to poor attitudes, negative attitudes lead to disengagement, and disengagement reduces success. But once the cycle is started it becomes self-reinforcing and self-perpetuating.

In contrast, positive student attitudes to mathematics, including confidence, a sense of self-efficacy, pleasure in and motivation towards mathematics lead to increased effort, persistence, and the choice of more demanding tasks. This is because of the intrinsic rewards such as intellectual satisfaction and the desire for success. The increased efforts and engagement in turn lead to students experiencing further success at mathematical tasks and mathematics overall. Consequently, positive student attitudes to mathematics are reinforced, completing a success cycle, in an enhancing upward spiral.

Psychologists, including Howe, (1990) have shown that a mechanism like that shown in Fig. 2 is an important factor in the development of exceptional abilities among gifted and talented students. Students who demonstrate some giftedness and talent at around the age of 10 are very significantly further ahead of their peers at the age of 20 precisely because of the factors shown in the figure. Early success and the attitudes it breeds lead to much greater effort, persistence, and choice of more demanding tasks which lead to the flowering of the later manifested exceptional abilities. Howe found that the exceptionally talented invested an extra 5,000 hours in practice of their skills and abilities. This was double the time spent by their capable but less outstanding peers. This finding has been popularized as the '10,000 hour rule' by Gladwell (2008).

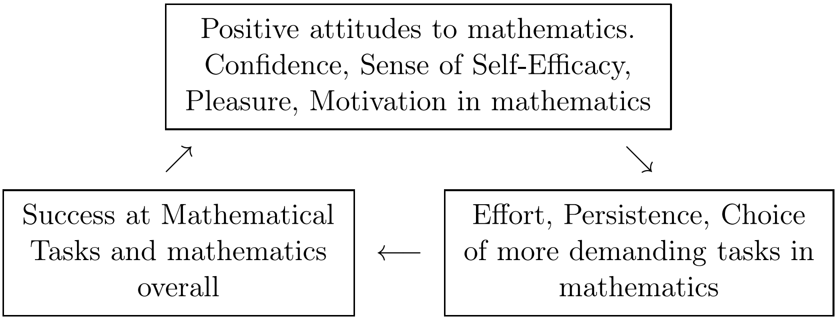

Figure 2. The Success Cycle (adapted from Ernest 2013).

Another impact of the social image of mathematics is in sex-differences. Traditionally females have had lower levels of achievement in school mathematics and lower levels of participation in advanced mathematical study and careers than males. Although school level achievements in mathematics have now balanced out, research shows that females continue to have, on average, more negative attitudes to mathematics than males, and this continues to be reflected in continuing lower levels of participation after the age of 16 years (Forgasz *et al.*, 2010). It is claimed that the social image of mathematics is a significant causal factor in these sex-differences (Mendick, 2006). Thus widespread gender stereotyped social images of mathematics include the view that mathematics is a male domain and is incompatible with femininity (Ernest, 1996). This contributes to gender stereotyped school images of mathematics which are manifested in a lack of equal opportunities, such as in classroom interactions in learning mathematics (Walkerdine, 1988; 1998). Social images as well as these school factors lead to gender-stereotyping in females' individual images of mathematics and impact negatively on their confidence and perceptions of their own mathematical abilities (Isaacson, 1989). The disadvantaging effects of these factors result in underachievement and lower participation rate in mathematics post-school. However, in the past two decades, female underachievement has been balanced out by male underachievement due to a separate set of factors, such as many young men's disengagement from school, especially in Anglophone countries such as United Kingdom (Forgasz *et al.*, 2010). However, rather than meaning that equality between the sexes has been achieved, it means that there are now two gender-rated problems related to school mathematics, and that these partially cancel out by negatively impacting differentially on both boys and girls. Furthermore, the lower female participation in higher mathematics post-school remains a significant problem.

Summary and provisional solutions

I have critiqued the idea that mathematics is an untrammelled force for good, the third public myth about mathematics. Instead I offer the metaphor that mathematics has two faces, the good and bad faces. The good face displays the benefits and value of mathematics. I have argued that mathematics is intrinsically a force for good, a creative development of the human spirit and imagination. It is also good in its utility, for it has many benefits in its social applications and personal value that benefit human flourishing. But, more controversially I also claim that mathematics has a bad face. It does harm through dehumanized thinking which fosters instrumentalism and ethics-free governance. Also, because of its over-valuation in the modern world through education it facilitates social reproduction and the perpetuation of class-based social injustice. Through its social image it aids the development of negative attitudes in some learners, and its gender-biased image maintains social disadvantage for females.

There are of course, in addition, ethically questionable and harmful applications of mathematics, as there are of any scientific and technological subject. Thus, for example, mathematics, science and technology are used in the manufacture of guns, explosives, nuclear and biological weapons, battlefield computer systems, tobacco products, and other potentially destructive artefacts and tools. But, there is a well known and legitimate argument that it is only in the choice of applications of mathematics in such activities that ethical considerations and violations emerge. My critique is independent of such deliberate applications, and perhaps even precedes them. I question and critique the views that mathematics is a unique and unified objective discipline (myth 1) that is ethically and socially neutral and value-free (myth 2), and that is solely a force for good incapable of detriment and social harm (myth 3) These views that I challenge, which I have termed myths hide the fact that mathematics through its actions on the mind is already implicated in some potentially harmful outcomes even before it is deliberately applied in social, scientific and technological applications.

However, some caveats to this argument are required. First of all, from the perspective that I term absolutist philosophies of mathematics, the image of mathematics that I have condemned follows as a necessary feature of mathematics emanating from its very nature. Although I and some others reject the associated absolutist epistemologies and ontologies these remain legitimate philosophies of mathematics. Secondly, the fact that the mindset fostered by mathematical thinking can lead to harm when it is misapplied to social and other philosophical issues is a defect of human or social thinking, and not an intrinsic weakness of mathematics. Thirdly, the damage done by social images of mathematics is mediated by interpretations of mathematics, that is socially and personally constructed images of mathematics. These images are not inescapable logical consequences of mathematics itself, for they can and have been different in different societies and different historical times. Thus the force of my critique is not directed at mathematics itself, but at the social institutions of mathematics, including training in mathematics, and the false social images of mathematics that they legitimate and project. The harm that I am highlighting comes from what are largely unconscious misapplications of mathematics, including the modes of thought it generates, and from the image of mathematics that many find excluding and off-putting, as well as the overvaluation of mathematical achievement in school and society.

Thus mathematics is not intrinsically bad or harmful, but as I have argued, its applications, both conscious and unconscious can be detrimental to many. This provokes the question: how can we prevent, ameliorate, or rectify this? In the space here I can only sketch a few possibilities for addressing these problems. My two main proposals are that we should include, in the teaching mathematics at all levels from school to university, (1) elements of the philosophy of mathematics and (2) the ethics of mathematics and its social responsibility

1. Teaching the philosophy of mathematics

My argument is that we should include selected aspects of the philosophy of mathematics in the school mathematics curriculum and in university mathematics degree courses. Students at all levels should have some idea of proof and how mathematical knowledge is validated. This includes knowing that no finite number of examples can prove a generalisation, whereas a single counterexample can falsify it. Students need to understand the limits of mathematical knowledge, including the following: the certainties of mathematics do not apply to the world, there is always a margin of error in any measurement; no mathematical application or scientific theory can ever be proved true with certainty, and this applies to any mathematical model of the world. Likewise we need to teach the limits of mathematical thinking: the true/false dichotomies we find in mathematics do not apply to the world, matters are almost never so clear cut. In addition, students need to be aware that there are controversies in the philosophy of mathematics over the nature of mathematics, the basis and status of mathematical knowledge and mathematical objects; that there are controversies over whether mathematical knowledge is absolute, superhuman with an existence that predates humanity, and over whether the objects of mathematics exist in a superhuman Platonic space. A live issue concerns whether humanly unsurveyable computer proofs, such as that of the 4-colour theorem are indeed legitimate proofs. Strong disagreements rage over whether mathematics is intrinsically value- and ethics-free or value laden, and over whether it is invented or discovered. I believe that elements of the history of mathematics and mathematics in history can serve to make some of the above recommended points and to humanize mathematics. This can be reinforced by illustrating the ubiquity of mathematics in culture, art and social life. Overall, my proposal is that students should see mathematics as more than just a set of tools, but instead be shown that it is long-standing discipline with its own philosophical issues and controversies, including human and ethical dilemmas, as well the nature and validity of its knowledge.

2. Teaching the ethics and social responsibility of mathematics

Although there is a widespread misperception that mathematics is neutral and bears no social responsibility clearly its uses and applications are value-laden. We should, in my view, add the ethics of mathematics to all university mathematics degree courses so that mathematicians gain a sense of its social responsibility. We need to teach that mathematics must be applied responsibly and with awareness, and that it is wrong to ignore or label its negative social impacts as ‘incidental’ outcomes or as ‘collateral damage’, and permit them to be viewed as outside of the responsibilities of mathematicians. In addition to teaching the ethics of explicit mathematical applications we also need to teach that mathematics has unintended ethical consequences. Thus, we need to teach the limits and dangers of instrumental thinking which mathematics can foster, and how it can lead to dehumanized perspectives in which people are both viewed and treated as objects.

Part of its social responsibility is to foster the public understanding of mathematics. Mathematicians, and more widely the professional mathematics community have the responsibility to promote the understanding of mathematics and to counter misconceptions and misunderstandings about the meanings and significance of the uses and applications of mathematics made public, especially in the media. Modern citizens should be critically numerate, able to understand the everyday uses of mathematics in society. As citizens, they need to be able to interpret and critique the uses of mathematics in social, commercial and even political claims in advertisements, newspaper and other media presentations, published reports, and so on. Mathematical knowledge needs to be critical in the sense that citizens can understand the limits of validity of uses of mathematics, what decisions are conveyed or concealed within mathematical applications, and to question and reject spurious or misleading claims made to look authoritative through the use of mathematics. Citizens need to be able to scrutinize financial sector and government systems and procedures for objectivity, correctness and hidden assumptions. Ideally they should able to identify the ethical implications of applications of mathematics to guard against the instrumentalism and dehumanization that often accompany technical decisions. My claim is that every citizen needs these capabilities to defend democracy and the values of humanistic and civilised societies, and it is part of the social responsibility of mathematics to help provide them.

A purist objection to such activities is, first of all, that they would steal valuable time and thus detract from the teaching of mathematics and second that these are not the responsibilities of mathematicians. With respect to the first objection it can be said that what I am proposing is not intended to take up even 2% of the time devoted to mathematics teaching in schools and universities. At school, such issues can be brought up within the mathematics curriculum periodically but without taking even a whole lesson. A discussion of examples, models and applications can lead to the issues being raised ‘naturally’, provided mathematics teachers have been well prepared to do this. At university a small, time limited course could be added as a mandatory course alongside pure, applied or service courses in mathematics. Thus this objection can be met, the costs in time could be very small, although the positive impacts in terms of mathematicians’ and other mathematics users’ awareness of the social responsibility of mathematics, could be significant.

With regard to the second objection it is first interesting to contrast the received views about the responsibilities of mathematics and mathematicians with parallel views about the social responsibilities of science and scientists. Unlike the case in mathematics, there is widespread acknowledgement of the social responsibility of science. Many have argued that what they term the Promethean power of modern science and technology warrants an extended ethic of social responsibility on the part of the scientists and technologists (Bunge, 1977; Cournand, 1977; Jonas, 1985; Lenk, 1983; Luppicini, 2008; Moor, 2005; Sakharov, 1981; Weinberg, 1978; Ziman, 1998). In particular, The Russell-Einstein Manifesto called for scientists to take responsibility for developing weapons of mass destruction and urged them to “Remember your humanity, and forget the rest” (Russell and Einstein, 1955). This manifesto initiated the Pugwash meetings which emphasised “the moral duty of the scientist to be concerned with the ethical consequences of his (sic) discoveries.” (Khan, 1988, p. 258). When accepting The Nobel Peace Prize on behalf of himself and the Pugwash conferences Joseph Rotblat stated "The time has come to formulate guidelines for the ethical conduct of scientist, perhaps in the form of a voluntary Hippocratic Oath. This would be particularly valuable for young scientists when they embark on a scientific career." (Rotblat, 1995). Thus Rotblat and his colleagues propose that ethics needs to be included in the training of young scientists, a call that is echoed by many others including Bird (2014), Evers (2001) and Frazer and Kornhauser (1986). This call has been taken up authoritatively by UNESCO which emphasizes the theme "Ethics of Science and Technology" (UNESCO, n. d.), and according to which "The ethics and responsibility of science should be an integral part of the education and training of all scientists". (UNESCO, 1999, section 3.2.71). Ziman claims that what is needed is what he calls 'metascience', an educational discipline extending "beyond conventional philosophy and ethics to include the social and humanistic aspects of the scientific enterprise" (Ziman, 2001, p. 165). He argues that metascience should become an integral part of scientific training in order to help equip scientists of the future with the skills necessary to tackle ethical dilemmas as they arise (Small, 2011).

The situation is rather different in mathematics with the exception of the Radical Statistics group (n. d.), which publishes analyses of social problem topics with the aim of demystifying technical language and promoting the public good. Generally, very few mathematicians acknowledge the ethical and social responsibilities of mathematics, although there is some acknowledgement of the social responsibility of mathematicians. Hersh (1990, 2007) discusses ethics for mathematicians Davis (1988) proposes a Hippocratic oath and the American Mathematical Society (2005) provides Ethical Guidelines for mathematicians. However, the content of these recommendations is primarily about professional conduct in research and teaching for professional mathematicians. Davis (1988) goes beyond this and argues that mathematics should not be put in the service of war or other harmful applications, and mathematicians should exercise their consciences. Ernest (1998, 2007) and Davis (2007) argue that mathematics needs to acknowledge its social responsibility, with Davis (2007) arguing for the need for ethical training throughout schooling for mathematicians and non-mathematicians alike. These, however, represent marginal voices in the mathematical and philosophical communities of scholars.

If one looks beyond mathematicians and philosophers to the area of mathematics education, there are many voices asserting the social responsibility of mathematics. Of course it is uncontroversial to claim that education is a value-laden and ethical activity, since it concerns the welfare of students and society, and the objectivity, purity and neutrality of mathematics itself is not at stake. In consequence, there is a very large literature comprising many thousands of publications on social justice and social responsibility in mathematics teaching.[2] Some of the main themes in this literature are mathematics and exclusion based on race and ethnic background (Powell and Frankenstein, 1997), gender and female disadvantage (Rogers and Kaiser, 1995; Walkerdine 1988; 1998), low 'ability' and handicap as obstacles (Ernest, 2011) and disadvantages correlated with or caused by social class and its correlated cultural capital or other factors (Cooper and Dunne, 2000). Another theme is the role mathematics plays in critical citizenship and the public understanding of mathematics (Frankenstein, 1990). A third theme is the Mathematics Education and Society (Mukhopadhyay and Greer, 2015), Critical Mathematics Education (Skovsmose, 1994; Ernest et al., 2016) and Ethnomathematics (D'Ambrosio, 1998) movements which consider both the role mathematics plays in society and how it impacts on the first two themes. The Critical Mathematics Education movement also looks critically at mathematical knowledge and the institutions of mathematics and their role in denying the relevance of ethics and values to mathematics, and thus denying its social responsibility (Skovsmose, 1994; Ernest et al., 2016). It shares this concern with the Philosophy of Mathematics Education movement (Ernest, 1991, n. d.), if such can be said to exist, to which this paper represents a contribution. However, within the mathematics education research community, beyond any commitment to the teaching of mathematics in a socially just way, the idea that ethics needs to be taught alongside mathematics remains a minority opinion, except perhaps within research in the third theme distinguished here.

Conclusion

In this paper I question and challenge three interconnected ideas about mathematics. First, I reject the idea that mathematics is a unique and unified subject, arguing instead that it comprises a set of overlapping epistemological knowledge domains and diverse social practices from research mathematics to applied mathematics, school mathematics everyday mathematics and ethnomathematics. Second I challenge the idea that mathematics is absolute, universal value-free and ethics-free. This challenge follows from a critique of the traditional separation of epistemology

2. A very partial bibliography of mathematics education published 20 years ago has over 800 mathematics education entries concerning the issues of society and diversity (Ernest, 1996).

and values, and links in with the first idea. Third, I argue that mathematics is not an unqualified force for good. I acknowledge the traditional argument that like any other instrument mathematics can be applied in both helpful and harmful ways, and I acknowledge the many benefits it brings. But I nevertheless endorse the minority view that mathematicians and other students of mathematics need to be taught the ethics of mathematical applications to question and limit harmful applications. My main argument, however, is more radical. I argue that in addition to the explicit and intended applications of mathematics, the nature of mathematical thought and the role mathematics plays in education and society lead to some presumably unintended but nevertheless harmful consequences. Mathematics has a hidden role in shaping our thought and society that is rarely scrutinised for its social effects and impacts, some of which are negative.

First of all, there is the harm caused by the overvaluation of mathematics in society and education, with its negative impacts on the confidence and self-esteem of groups of student including females and lower attainers in mathematics. These unintended outcomes of mathematics in school leave some students feeling inhibited, belittled or rejected by mathematics and perhaps even rejected by the educational system and society. In sorting and labelling learners and citizens in modern society, mathematics reduces the life chances of those labelled as mathematical failures or rejects (Ruthven, 1987). This is a hidden impact of mathematics that is usually brushed over as the fault of the individuals that suffer, rather than as a direct responsibility of the role accorded to mathematics in education and society.

Second, even for those successful in mathematics, in shaping thought in an amoral or ethics-free way, mathematics supports instrumentalism and ethics-free governance. Instrumental thinking leading to the objectification and dehumanisation of persons in business society and politics has the potential to cause great hurt and harm. This is manifested, in warfare, the actions of psychopathic corporations, the exploitation of humans and the environment, and in all acts that treat persons as objects rather than moral beings deserving respectful and dignified treatment throughout (Marcuse, 1964).

I do not claim that mathematics is intrinsically harmful, but that without more careful thought about its role in society it leads to harmful, albeit unintended, outcomes. My proposal is that to obviate or prevent the potential harm done by mathematics we need to teach the philosophy and especially the ethics of mathematics alongside mathematics itself. Part of this teaching is needed to overcome the myths that I have challenged here, especially the idea that mathematics, unlike any other domain of human knowledge bears no social responsibility for its roles in society, science and technology. All human activities should contribute to the enhancement of human life and general well-being and no domain can stand apart from such ethical scrutiny, although this should never be used as a reason for limiting advances within pure mathematics itself. However, the intended and unintended applications of mathematics and their consequences do need to be scrutinised and held accountable within the court of human happiness and human flourishing.

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Scientific thinking versus religious thinking from a view point of a secular science educator

Abstract. In my paper I'll characterize Science Education as a research discipline. I will characterize scientific thinking versus religious thinking. The main difference is that scientific truths are refutable whereas religious truths are irrefutable. Thus, relying on Popper, we can consider religious thinking as pseudo-scientific thinking. The talk will compare the contribution that science and religion can offer to face the main problems of human beings: mortality and suffering.

Science can hardly facilitate our dealing with the main problems of human beings: mortality and suffering. On the other hand, the solutions which religion offers to us have their own problematic.

As you can guess, I am quite old. And when people become old they become reflective, they become philosophical, they become skeptical. They ask questions about their life projects. I consider myself a civil servant in the field of education. I taught mathematics; I taught mathematics education and I taught science education.

The reason for this is that I consider mathematics as part of science. It is so because all sciences (including social sciences) use mathematics in order to develop their theory. And also, their structure is quite similar to the structure of mathematical theories, namely, the deductive structure.

In recent years I have asked myself what could be the contribution of my teaching to the well being of my students. My main concern was scientific thinking. On the other hand, there is a lot of religious thinking around us and I wonder what this thinking could contribute to believers. Thus, I have come to make the comparison between the two and hence

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Key words and phrases: scientific thinking, religious thinking, science education, Popper, pseudo-science, secular, atheist, Spinoza, pantheism, Euler, Hamlet, Diderot, God, transcendental God, immanent God..

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the title of my paper: “Scientific thinking versus religious thinking from a view point of a secular science educator”.

As an introduction I would like to elaborate on some notions in this title.

1. Scientific theory is a theory which explains phenomena in our world and in most cases is supported by experiments or facts. A scientific theory can be refuted (remember Popper). On the other side, there is also the notion of pseudo-science. A pseudo-science looks like a science, but essentially, it cannot be refuted.

   Some examples of pseudo-sciences are psycho-analysis, some parts of biological evolution and religious explanations to phenomena in our world (particularly, the creationist theory)
2. Religious thinking explains our world relying on religious texts. One of the most common explanations relies on the claim that things are the result of God’s will (the creator’s will).
3. Science Education is an ambiguous notion. On one hand it denotes the act of teaching science to students and related activities such as curriculum development, writing science textbooks, computer software etc. On the other hand it is a research discipline which investigates thought processes of students learning science and of teachers teaching science.
4. By the term “secular” I mean somebody who does not believe that God exists. For me the term “secular” has the same meaning as “atheist”. Unfortunately, the term “atheist” has a negative connotation. It is not politically correct. Therefore many atheists avoid declaring themselves as atheists. They prefer to appear as pantheists. Pantheism is the belief that the **universe** (the totality of everything) is identical with **God**. This idea is due to the Jewish philosopher, Baruch Spinoza (1632-1677).

   Nevertheless, since I believe that political correctness is very often the enemy of truth, I will use “atheism” in this paper just to clarify the distinction between religious people and secular people.
5. The view point of science educator in this talk is the view point of somebody who tries to understand the thinking of religious people.
6. The religions which I will discuss are Judaism and Christianity. From time to time, while being involved in theological discussions with my gentile friends, I remind them that Jesus was Jewish and was strongly influenced by the Jewish conception of life and the world. Other religions which I hardly know, like Islam and Far East religions will be mentioned shortly and superficially.

7. Science education is interdisciplinary. It borrows various tools from cognitive psychology, the psychology of problem solving, sociology, philosophy of science and theology. It is not supposed to make innovations in those disciplines. Thus, in this paper I am borrowing from those disciplines; I am not intending to make any innovations.

The purpose of this talk is to explain how scientific thinking as well as religious thinking can help us to cope with our daily problems, physical and psychological. In my opinion our main problems are health, mortality and suffering.

I'll start with scientific thought. Scientific thought offers us, among other things, ways to cope with our health problems. It includes ways to deal with our physical pain. Scientific thought also offers us explanations about our physical world from the moment it was created till present. It gave us the magnificent edifice of natural sciences which is also the basis for medicine and technology. However, science does not help us to cope with our psychological problems mentioned above: mortality and suffering.

My main source for religious thinking will be the Bible which also explains how the world was created and how all what we see here came into being. So, let's start with the religious explanation to the creation, namely, with Genesis, chapter 1.

At the end of six day work God was quite satisfied. "God saw everything that he had made and, behold, it was very good" (Genesis, 1, 31).

As a matter of fact, God was the only one who was satisfied. Adam and Eve were not cognitively ready to make any evaluation at this stage. Why? Because they had not eaten from the tree of knowledge of good and evil. At this point, they were not aware of their mortality. God was aware. But God did not think it was negative. However, Adam and Eve realized their mortality after eating from the tree of knowledge and they thought it was horrible.

This takes me directly to Ecclesiastes. It is the most pessimistic book which I know. It is pessimistic because of the writer's understanding that he is mortal. If I have to pick up a typical quotation from Ecclesiastes I will choose chapter 9, 4-6. "A living dog is better off than a dead lion! The livings know that they will die, but the dead know nothing; their love, their hate and their jealousy have long since vanished."

The mortality situation leads us to the following million dollar question: If we eventually die why are we born for? And if we are born what is the point of ending our life?

Is there a way to solve this dilemma? Well, there is a simple solution. It is based on the distinction between body and soul.

It is quite interesting that there is a hint to this distinction in Ecclesiastes. “And the dust returns to the ground it came from, and the spirit returns to God who gave it” (Chapter 12, 7).

However, Ecclesiastes himself, probably, was not convinced. If I am allowed, I would like to make a wild assumption here. Ecclesiastes has a realistic way of thinking. This is in a way, a scientific way of thinking. Namely, thinking which considers facts. The idea that there is a place to which souls go after the death of the body has no support in our world. Thus if you don’t accept this idea, what is the answer to the above million dollar question?

The answer is quite pessimistic: Without being asked we are born and eventually we have to die. Can we carry on our life with this message? For many of us it is unacceptable. This leads us to the search for the meaning of life. Here is an accidental list of 5 references which deal with this question:

The Meaning of Life (Klemke, 1981);
The meaning of it all (Feynman, 1998);
Man’s Search for Meaning (Frankl, 1959);
Momma and the meaning of life (Yalom, 1999);
The Meaning of Life (Eagleton, 2007).

Here is a quotation from a review of Eagleton’s book.

> How can an English professor and literary critic (that’s Eagleton) write a philosophical brief on the meaning of life? Well, Terry Eagleton did, and did it well.
>
> He takes us through the end of Victorian certainty and shows how Hardy and Conrad raised questions with a sense of urgency that Jane Austin never had. In the early decades of the 20th Century, T. S. Eliot and Camus and Sartre brought challenges to all our values, beliefs and institutions.
>
> Most in the West have now accepted the view that life is an accidental evolutionary phenomenon with no intrinsic meaning. Rather than lament the loss of being part of God’s design, which was often impenetrable, this clears the ground for us to give life meaning whatever we choose.
>
> A starting point is realizing that life is not a problem to be solved; if we are being practical, it really becomes more an ethical issue than metaphysical. We should be more concerned about what makes life worth living, what adds quality, depth, abundance, and intensity. Eagleton’s suggestions point us towards a direction we have heard before, caring for others, compassion, becoming truly engaged. And, that is what has occupied the great novelists, poets and artists of all ages.

Unfortunately, some people do not want to follow Eagleton's altruistic recommendations. Their way to cope with the evolutionary claim that life is meaningless by denying it. This is easily done by drinking alcohol, by smoking grass and using other drugs. Li-Tai-Po (701-762), a Chinese poet whose poems were used by the Czech Jewish composer Gustav Mahler, writes:

Wine is already beckoning in the golden goblet,
But do not drink yet:
First I'll sing you a song!
The song of grief shall sound,
Laughing, in your soul:
When sorrow approaches, the gardens of the soul lie withered,
Joy and song fade and die.
Dark is life, dark is death.

Thus if life is only suffering, what meaning does it have? Here is another question about the meaning of life taken from the famous movie "Hair":

> Wine is already beckoning in the golden goblet,
> But do not drink yet:
> First I'll sing you a song! The song of grief shall sound,
> Laughing, in your soul:
> When sorrow approaches, the gardens of the soul lie withered,
> Joy and song fade and die.
> Dark is life, dark is death.

However, many people make a real effort to find a meaning to their life. Here are three examples.

The first one is a scene from Woody Allen's movie "Hanna and her sisters". Allen, after his medical doctor told him he did not have brain cancer, realizes that he is mortal. He returns to his secretary and tells her about his discovery and about his need to find a meaning to his life. Allen is denoted by A and the secretary is denoted by S. The following dialogue takes place:

A: Do you realize on what a thread our life is hanging by?
S: Micky, you are off the hook. You should be celebrating.
A: Do you understand how meaningless is everything, everything I am talking about? Our life, the show, the whole world are meaningless.
S: But you are not dying.

A: Now! I am not dying now, but. . . Do you know? When I ran out of the hospital I was so thrilled because they told me I was going to be alright. But being on the street, suddenly I stopped because it hit me: I won't die today, I am OK. I am not going to die tomorrow, but eventually, I am going to be in this position.

S: Are you realizing it just now?

A: I do not realize it just now. I knew it all the time, but I managed to stick it in the back of my mind, because it is very horrible thing to think about. Can I tell you something? Can I tell you a secret?

S: Please!

A: A week ago I bought a rifle. Do you understand? I bought a rifle because if they would have told me I have a tumor I was going to kill myself. The only thing that might have stopped me is my parents. I would have to shoot them also first and then my uncle and my aunt, bloodshed.

S: Eventually, this is going to happen to all of us.

A: Yes, but doesn't it ruin everything for you? Doesn't this take the pleasure of everything? I am going to die, you are going to die, the audiences are going to die, the net, the sponsor...

S: I know, and your hamster. Listen, I think you snapped a cap. Go to few weeks in Bermudas or go to a hoar.

A: I can't stand the show. I have to get some answers.

Allen's solution to his existential problem is looking for God within the walls of the Catholic Church. The following dialogue takes place at the priest office. The priest is denoted by P and Allen is denoted by A.

P: Well, why do you think you have to convert to Catholicism?

A: Well, because, you know, I have to believe in something, otherwise life is just meaningless.

P: I understand, but why did you make the decision to choose to the catholic faith?

A: You know, first of all it is a very beautiful religion, it is a strong religion and it is very well structured. I am talking now against school prayers, anti-abortion and nuclear war.

P: But at the moment you don't believe in God. . .

A: No, and I want to. I am ready to do everything. I am ready to dye Easter eggs if it works. I need some evidence. I have to get some proof. If I don't believe in God I think life is not worth living.

P: It means a very big leap.

A: can you help me?

A relatively easy way to deal with the mortality problem is the idea of the next world, or in Jesus' terminology, the heavenly kingdom. However, the claim about the next world is a pseudo-scientific claim. Namely, according to Popper's criterion, it cannot be refuted. Probably, this is the reason why there are so many people who are ready to buy it. It brings some relief to their fear to die.

Beside the fear to die there is another problem for many people, which is the suffering problem. Especially, the suffering of people who have tried to follow all the commands of God and are expecting to be rewarded by God for their good deeds. Thus, by inventing the Heavenly Kingdom, Jesus has killed two birds in one shot: He, supposedly, cured us from our fear to die. And also he promised us a huge compensation for our suffering. No wonder why so many people all over the world converted to Christianity. In order to overcome the danger that also Jews would adopt Christianity, the idea of the next world was adopted also by Judaism. Note that the idea of the next world does not exist in the Jewish Bible (the Old Testament).

Now, the existential situation of human beings is the following: We hate dying but we also hate suffering. However, ironically enough, death can liberate us from our suffering. The question is quite simple: which emotion is unbearable? Our fear to die or our suffering?

Hence, when suffering overcomes the fear to die suicide is recommended. Here are some examples from the Jewish history:

1. King Saul (1 Samuel, Chapter 31) committed suicide when he realized that he lost the battle with the Philistines and they were going to capture him.
2. Ahitophel (2 Samuel, chapter17) committed suicide after his military recommendation was not accepted by Absalom in his rebellion against King David. Ahitophel could not bear the humiliation and he hanged himself.
3. The defenders on Mount Masada and their families committed a collective suicide when they realized the Romans were going to win the battle and as a result they would be taken to Rome as slaves.

The suicide option is also suggested by Shakespeare in Hamlet's ultimate monologue:

To be, or not to be–that is the question:
Whether ’tis nobler in the mind to suffer
The slings and arrows of outrageous fortune
Or to take arms against a sea of troubles
And by opposing end them. To die, to sleep;
No more; and by a sleep to say we end
The heartache, and the thousand natural shocks
That flesh is heir to, ’tis a consummation
Devoutly to be wished. To die, to sleep;
To sleep: perchance to dream: ay, there’s the rub,
For in that sleep of death what dreams may come
When we have shuffled off this mortal coil,
Must give us pause. There’s the respect
That makes calamity of so long life.
For who would bear the whips and scorns of time,
The oppressor’s wrong, the proud man’s contumely
The pangs of despised love, the law’s delay,
The insolence of office, and the spurns
That patient merit of the unworthy takes
When he himself might his quietus make
With a bare bodkin? Who would fardels bear,
To grunt and sweat under a weary life,
But that the dread of something after death,
The undiscovered country, from whose bourn
No traveler returns, puzzles the will,
And makes us rather bear those ills we have
Than fly to others that we know not of?
Thus conscience does make cowards of us all,
And thus the native hue of resolution
Is sicklied o’er with the pale cast of thought,
And enterprises of great pitch and moment
With this regard their currents turn awry
And lose the name of action.

It is interesting that Shakespeare does not mention diseases as a reason for suicide. We can wonder why that is. Is it because Shakespeare was not aware of diseases or because of literary reasons? After all, Shakespeare did not write an academic essay about reasons to commit suicide. He wrote a play. Nowadays, many terminal patients prefer to die than to go on with their terrible suffering. However, being so helpless, these patients cannot commit suicide by themselves. They need help. But only in few States all over the world it is legal to help people to commit suicide. This is only a by the way comment. Hamlet did not commit suicide because he wanted to find out what caused his father's death. Among the reasons which justify suicide in Hamlet's monologue there is despised love. Unfortunately, it is demonstrated by Ophelia in Act IV, Scene 5. Hundred and seventy years later, also Werter, the hero of Goethe, as well as Anna Karenina, Tolstoy's heroine, and Madame Bovary, Flaubert's heroine, committed suicide because of despised love.

Before moving on to the next issue I would like to remind you of two notions from the science of religions. The first one is Transcendental God.

The transcendental God is the God who created the world and after that he stopped being involved in events in our world. The concept of transcendental God and Pantheism are almost the same in my opinion.

The second notion from the science of religions that I would like to use is the notion of Immanent God. The immanent God is supposed to interfere with events in our world. Namely, He is supposed to reward the people who follow his commands and to punish the sinners.

The situation of somebody who is convinced that he did not do anything wrong and in spite of that his suffering is unbearable is quite typical. There are two Biblical figures that raised this problem protesting against God: Jeremiah and Job. In chapter 12, 1 of Jeremiah he says: "You are always righteous, Lord, when I bring a case before you. Yet I would speak with you about your justice: Why does the way of the wicked prosper? Why do all the faithless live at ease?" Jeremiah preferred to pose his question in a general way. He did not mention that he was the one who was persecuted by the wicked. On the other hand, Job's complaint is personal. He wanted to know why he was suffering. Also for him death was suggested as a solution to his suffering. It was suggested by his wife. "Curse God and die" (Chapter 2, 9), she advised. Job refuses. But his general mood is the same as the mood of the prophet Jonah (chapter 4, 30): "It is better for me to die than to live". In chapter 3, 11-13 Job says: "Why died I not from the womb? Why did I not give up the ghost when I came out of the belly? Why did the knees prevent me? Or why the breasts that I should suck? For now should I have lain still and been quiet, I should have slept: then had I been at rest" (King James translation).

A similar mood is expressed in Psalm, chapter 22: "My God, my God, why hast thou forsaken me? I am a worm, . . . and no man; . . . For dogs have compassed me: The assembly of the wicked has inclosed me: **they**

pierced my hands and my feet." By the way, this last sentence does not exist in the Hebrew version of Psalm 22, nor in the English Version which was made by the Gideons.

My assumption is that king James' translator, being aware that Jesus referred to Psalm 22 while suffering on the cross, and being a good Christian, he decided to add this sentence to the text because it describes accurately the crucifixion. And for Jesus, feeling he was abandoned by God, it was only natural to chose Psalm 22 to express his unbearable pain on the cross. Indeed, "My God, my God, why hast thou forsaken me?" were Jesus' last words on the cross before he died.

A possible answer to Jeremiah's question as well as to Job's question is that God moves in mysterious ways. This is beautifully expressed by Job's words: "Therefore have I uttered that I understood not; things are too wonderful for me, which I knew not" (Chapter 42, 3). In the book of Job God relates to Job's complaint by the following:

"Where were you when I laid the earth's foundation?. . . Who marked off its dimensions?. . . Who shut up the sea behind doors?. . . Have you ever given orders to the morning or shown the dawn its place?. . . Have you comprehended the vast expanses of the earth?" (Job, chapter 38). If I try to formulate God's answer to Job's complaint in our daily language it will be something like: Who are you to question my ways of conducting the world?

Nevertheless, in case you are not satisfied with the way the book of Job is ended, and you still want to deal with the difficulties it poses to the idea of an immanent God, there is another way to cope with Job's tragedy. It is to claim that Job did not exist. It is only a story. But for me, as a secular reader, even if it is only a story, God doesn't come out so good from it. God bets with Satan that Job will not lose his faith, no matter what happens to him. Thus, he lets Satan destroy Job's entire property, he lets Satan kill Job's ten children and make him sick. Moreover, even if Job is only a story, we, unfortunately, have met and have heard of many people whose suffering was as bad as Job's suffering.

Moving from the biblical era to our time, the idea of immanent God faces enormous difficulties when considering the horrible events of genocides in Auschwitz and in other places all over the world. And again, the only way to justify it is by claiming that we, human beings, cannot understand the ways God operates our world.

The bottom line at this point of my paper is that the secular thinking is much simpler than the believers' thinking since it does not have to cope with all the above mentioned problems. This is not a call to believers to abandon their belief. Sometimes, when I discuss these issues with my students, some of them try to convince me that God exists. A common argument for Islamic students is that the Koran is so beautiful that only God can write such a text. Since this is claimed in a science education course I remind my students that in science we have to examine all the time alternative hypotheses. Thus, a simple alternative hypothesis to the last one is that some human beings have such a wonderful literary talent that they can write such a beautiful text as the Koran. This of course does not convince them. They stick to their belief and I stick to mine. Another proof for the existence of God is like the following: Consider things around you, furniture, buildings, cars etc. All of them were constructed by somebody. Is it possible that the entire world was not created by somebody? Unfortunately, also this proof is not valid. It is impossible to conclude from the fact that a table is constructed by a carpenter that also the world has a creator. Such conclusion is to assume what you want to prove.

In the history of the theological debates about the existence of God there is a story about a debate between Euler, the religious, and Diderot, the atheist. This debate was initiated by Catherine the Great (1729-1796), the Czar of Russia, a religious lady, who was disturbed by the French atheistic movement. She invited Diderot, a philosopher and one of the main leaders of the atheistic movement, to argue with Euler about the existence of God. Euler said: $[(a + bn)/x] = x$ therefore God exists. Diderot, whose mathematical knowledge was almost zero, realized he could not contradict Euler's argument and, therefore, he returned to Paris. This anecdote tells us something about Euler. First, in order to win the debate he did not avoid unfair means. But moreover, being a mathematician (one of the greatest in mathematical history), he knew that what he said could not be a proof that God exists. Namely, he believed in God not because there was a proof that God exists but because he chose to believe in Him. There are more proofs for the existence of God. For instance, God appeared to me in my dream. Again, to my science education students who claim it I say: Can you suggest an alternative interpretation to the fact that you have dreamt about God rather than the claim that God exists? Have you heard about Freud?

Thus, when I try to convince my students that there is no proof for the existence of God, some of them don't believe me. Others, who developed critical thinking, understand it. Is the belief in God weaker when somebody realizes that there is no proof for His existence? I really don't know.

However, in my opinion, the real question is not whether God exists. The real question is whether God is needed. And the answer to this question is Yes (with Capital Y). Everybody needs Him, the religious as well as the atheists. For me, as an atheist, he is a metaphor. But I am also inspired by literary texts in which God is the main figure. I am inspired by the Bible, and I am inspired by musical compositions as masses and requiems, some of which were composed by secular composers as Berlioz and Verdi.

By saying this I have almost reached the end of my paper. I would like to summarize it with the following comment:

As a secular science educator I have tried to compare religious thinking to scientific thinking about the essential aspects of life. I have revealed my own thinking. My claims about the thinking of religious people on these issues are mere speculation. Hence, I would like to invite religious science educators to react to this talk and to reveal their thinking about the above issues. You are invited to send me an email. My email address is: vinner@vms.huji.ac.il

And finally an apology: I apologize for involving you with an unpleasant topic – our mortality. However, we face death almost every day. We go to funerals, we write wills and so on and so forth. I believe that most of us cope with these facts by adopting Jan Francois Vilar's famous claim: *C'est toujours les autres qui meurent.* And in English: *It is always the other people who die.*

And with this happy note let us return to our homes and families and enjoy our life as much as we can.

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Contrat, transgressions et creation. Une tentative de clarifier les paradoxes de la relation didactique dans l'enseignement des mathématiques en utilisant une approche didactique et anthropologique[1]

Abstract. During the first part of the lecture, we will study, from a theoretical point of view, the issue of transgression as an expected response given by a pupil, but unrequired by the teacher. This phenomenon is the paradoxical result of the contract which forms during any didactical relationship: "this is what you have to know, and from now on, think for yourself to show that you are able to create new uses out of what you have been taught; in other words, act in accordance with what I have taught you, but don't obey me!". So, transgression will be considered as a necessary condition for learning mathematics (different from the use of techniques, algorithm, and rules) whose conditions of existence stand at the crossroads of determinations which are both didactical (with reference to "the paradox of devolution" as defined by Guy Brousseau in the theory of didactical situations) and anthropological (with reference to the concept of "use" in Wittgenstein's anthropology and to his famous rule-following paradox).

During the second part, we will base our argument on various research in order to underline:

a) the relevance and the interest of this theoretical approach in order to gain a better understanding of the reasons for pupils' and teachers' recurrent difficulties (for example, "you know the lesson, the teacher says, but you didn't understand it."),

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1. Contract, transgressions and creation. An attempt to clarify the paradoxes of the didactical relationship in mathematics education using a didactical and anthropological approach

and the reasons why some of the means intended to regulate these difficulties fail, and

b) the role of "backgrounds" (in the Searlian meaning of the word), such as familial educational practices and the didactical and pedagogical cultures of school environments (which are linked to values, beliefs, epistemological and pedagogical conceptions of the teachers) in order to account for the appearance of interindividual differences concerning the relations with transgression, and clarify the ways we can go beyond the initial paradox.

In conclusion, we will promote the idea of a "normative transgression" to describe this phenomenon of the sudden appearance of new creations ("transgressive" dimension) which are expected by the teacher and lived by the pupil as a measured disobedience, for it is basically in accordance with the "account books" of mathematicians (normative dimension). This is probably where the fascinating and singular essence of mathematical activity stands, between logical constraints and the boundless openness of creative possibilities.

Introduction

Le sens de «transgression » est très variable selon les cultures, les époques, les situations … à tel point qu'il est très difficile de savoir ce qu'il désigne. Je prendrai ici « transgression » dans son plein sens étymologique : est transgression toute action qui franchit, qui traverse, qui dépasse … une « limite » (à l'origine, il a donc un sens très proche de « transmettre » (faire passer au-delà, de l'autre côté) mais aussi « transcender » (monter en passant au-delà, dépasser, franchir …) ; ce n'est que bien plus tard que « transgression » prendra le sens de « désobéissance » ou de « violation ». Pour le dire rapidement, je situerai la transgression dans le rapport qu'un individu établit avec ce qu'il considère comme un représentant d'une institution (institution étant pris ici au sens d'une instance symbolique qui rend légitime une norme, une règle, une loi …).

La transgression n'est à confondre avec la désobéissance : je réserverai ce terme à toute l'action d'un individu qui viole *intentionnellement* un interdit posé par un autre individu.

Il n'y a pas de désobéissance sans une intention : celle de ne pas respecter un interdit : la désobéissance suppose donc la connaissance de l'interdit (les enfants ne s'y trompent pas lorsque après avoir fait une bêtise se réfugient dans l'ignorance : « Je ne les avais pas ! »).

FIGURE 1.

Par exemple, il n'est correct pas de dire que le renard « vole » ces gâteaux, qu'il désobéit à l'interdiction de s'approprier sans autorisation le bien d'autrui. Il les prend simplement.

Ainsi, on dira qu'on *transgresse* une norme, une loi ... mais qu'on *désobéit* à une personne en ne respectant pas l'interdit qu'elle a posé.

Cette distinction n'a aucune prétention universelle ou historique ; elle a seulement un intérêt pragmatique : elle permet de préciser le sens dans lequel j'utiliserai ces termes dans la conférence. Par exemple, désormais vous pouvez comprendre le sens dans lequel j'emploie ces termes dans les phrases suivantes :

- « Eve désobéit à Dieu mais transgresse la loi dont il est porteur »
- « Prométhée est puni pour avoir transgressé les lois divines, mais pas pour avoir désobéi à Zeuset pour avoir transmis le savoir aux Hommes »

La deuxième remarque concerne la nature de la limitede la transgression.

Limite peut-être employée en deux sens :

1. Au sens d'une borne infranchissable, vers laquelle on ne peut que tendre. Par exemple : 299792458m/s est la vitesse limite pour les particules qui ont une masse nulle, etc.
2. Prise dans le sens de frontière, une limite borne alors deux espaces distincts ; elle marque la frontière entre deux espaces, entre deux territoires, entre deux états … Par exemple, il est facile d'imaginer une situation où il faudrait tuer quelqu'un pour en sauver plusieurs autres et pourtant « Le commandement éthique dit : tu ne tueras point. Il ne dit pas : tu ne tueras point, *sauf si* … Il dit : tu ne tueras point, *point.* » (cf. Castoriadis,1996 ; Rorty, 1993, 16).

Il convient d'être extrêmement attentifs à ne pas amalgamer les deux usages car la première est indépassable et donc ne peut être transgressée car l'interdit n'a aucun sens mais seulement une impossibilité ; alors que le second sens, la limite est non seulement contingente, mais elle est aussi variable selon les époques, les lieux, les cultures …

La transgression atteste de la dimension factice de la limite et de l'arbitraire de l'interdit ; elle montre bien que la limite n'en était pas une au sens 1 (de l'impossibilité) puisqu'elle s'est avérée franchissable ; *la transgression transforme donc les limites en frontières.*

C'est en cela que la transgression est toujours démesure, dépassement … et ce faisant, elle permet d'interroger après-coup la validité et la légitimité … des interdits qu'elle posait. La possibilité (comme éventualité) de la transgression pose donc le sujet comme responsable de sa décision : il est libre de la prendre ou de la rejeter : en cela, la transgression n'est pas immorale en tant qu'elle ne maltraite pas les normes, mais par le dépassement qu'elle opère, par sa démesure, la transgression transforme les normes mais sans les détruire.

La transgression dénaturalise les normes en révélant leur caractère construit. Doit-on rappeler ici le supplice infligé par les Chrétiens à cette philosophe et mathématicienne grecque Hypatie (370-415), qui eut simplement tord d'avoir raison. Nicolas Copernic n'a-t-il pas refusé de publier son œuvre majeure durant sa vie ? Eve transgresse la loi, désobéit à Dieu, mais ce faisant marque l'existence du libre-arbitre : c'est paradoxalement l'interdit qui crée la condition de notre liberté : c'est d'ailleurs particulièrement flagrant dans le cas des mathématiques car nul autre secteur de la culture pose des contraintes aussi formelles et aussi puissantes qu'en mathématiques : c'est d'ailleurs une des raisons pour laquelle elles constituent un espace infini de créations !

Le troisième repère est celui de la contextualisation de la question de la transgression dans le champ de l'éducation mathématique. *A priori*, la transgression n'a aucune place dans champ de l'éducation puisque l'éducation est le processus de transmission des normes, valeurs et des savoirs. En effet, comment peut-on à la fois vouloir transmettre et faire une place à transgression dans ce processus ? La réponse à cette question occupera la première partie de ma conférence mais je poserai dès maintenant un troisième repère. « Apprendre », comme « aimer » sont des verbes qui ne supportent pas l'impératif ! L'élève ne peut jamais être *contraint* d'apprendre ou de comprendre. Et pourtant on exige de lui à la fois obéissance (« Fais ceci, retiens cela ... ») [2] mais en même temps on exige aussi qu'il raisonne par lui-même. C'est bien cette double exigence que l'on retrouve chez Kant quant il articule les deux types d'usage de la raison : l'*usage privé* correspondant à celui que doit en faire l'homme *librement* en tant qu'individu, et l'*usage public* par lequel l'homme doit être en accord avec les règlements de la communauté à laquelle il appartient. Autrement dit, dans le champ même du savoir, la seule limite que je puisse rencontrer c'est celle du désir, du désir de savoir ! C'est précisément ainsi que Piera Aulagnier définit la transgression : elle est ce qui réalise « cette visée du désir de ne rencontrer aucune frontière à son champ d'action, quoi qu'il sache et quel que soit l'infini de son désir de connaître, il ne manquera jamais d'objet à interroger ni un dernier voile à soulever. Le manque à savoir est ce qui ne manquera jamais à son désir » (Aulagnier 1967). Autrement dit, on en a jamais assez de savoir car, comme l'a bien montré Freud, Lacan ... ce désir de savoir se nourrit de « l'énergie du plaisir scopique » (Freud, 1987, 123), et conduit à maîtriser les incertitudes attachées aux situations (c'est ce montre Freud dans le Fort-da). C'est une des raisons pour laquelle le savoir à toujours partie liée avec l'interdit (Le pêché originel, Umberto Ecco Le nom de la rose) et donc avec la transgression.

1. Transgression et création

Si une société, un collectif, une famille ... ne peuvent transmettre que l'existant, alors toute création exige une transgressionpuisqu'il s'agit

2. Les élèves sont toujours tenus de fournir des indices de leur volonté d'adhésion au projet d'éducation ; comme le dit Weber de toute personne qui appartient par naissance et par éducation à une institution « on attend d'elle qu'elle participe à l'activité communautaire et tout particulièrement qu'elle oriente son activité d'après les règlements, et en moyenne on est en droit d'attendre cela d'elle, parce que l'on estime que les individus en question sont 'obligés' empiriquement de prendre part à l'activité communautaire constitutive de la communauté et qu'on y rencontre la chance qu'ils sont tenus de le faire sous la pression d'un appareil de contrainte (si douce que soit sa forme), éventuellement mȩme contre leur gré. » (Weber, 1992, 353).

de faire exister l'inexistant, de rendre visible le caché. Or, la psychanalyse montre bien que le caché a toujours entretenu un commerce subtil avec le savoir et le désir : le sujet ne peut désirer que ce qu'il n'a pas, ce qui se dérobe à lui, ce qui lui est caché. Tel est là un des moteurs puissant de la création en quête de savoir : un savoir pour voir ! Le caché a conséquemment toujours été frappé d'interdit : souvenons-nous de la Genèse de la Bible, l'arbre de la connaissance, ou encore de ce beau roman de Umberto Eco, *Le nom de la rose* ; songeons aussi aux enfants, à leur attirance particulière pour l'interdit ou le caché ... c'est dans cet espace que naît ce « désir de voir » que Freud appelait « *pulsion scopique* » (1987, 123). Ainsi la création est toujours transgression : transgression des normes pour voir autrement, transgression des interdits en s'autorisant soi-même à voir ce qui est masqué. Or, la transgression – et donc la création– ne peut être le produit d'une injonction, l'exécution docile d'un ordre : Désobéissez ! Thomas Khun (1983) parle à cet égard des « révolutions de scientifiques », « révolution » au sens de considérer les choses autrement, sous un autre angle ; ce fut par exemple le cas de Andrew Wiles qui considéra la conjecture de Fermat à partir des travaux de Galois et de Taniyama-Shimura (Singh, 1998). Tel est le troisième paradoxe : créer, c'est s'autoriser, c'est décider de devenir auteur, c'est-à-dire adopter la posture de « celui qui fonde et établit » – en latin *auctor* désignait Dieu, un Dieu créateur. Mais s'autorisersuppose que le sujet soit autonome et libre mais, on l'a vu, cette liberté n'est possible que dans le cadre d'une culture nécessairement collective. Ce paradoxe trouve son expression philosophique dans la célèbre formulede Kantcaractéristique de l'esprit des Lumières : « Raisonnez autant que vous voulez et sur ce que vous voulez ; mais obéissez ! » (1991, 50). Un joueur de hockey, un peintre, un musicien comme un mathématicien, est libre dans sa manière de jouer mais pas dans la définition du jeu auquel il participe ; sa liberté de joueur n'est possible que s'il s'assujettit aux règles qui fixent les conditions de possibilité de son jeu au sein d'une communauté (celle des mathématiciens, des hockeyeurs, des cuisiniers ou romanciers ...). C'est dans cet entre-deux, entre la liberté individuelle et les contraintes collectives propres au jeu, entre la dimension structurelle du jeu et la manière de jouer que se situe l'espace de la création. Mais répétons-le, la création est loin de se réduire à la nouveauté, elle doit aussi présenter un intérêt, une valeur pour l'institution ou la communauté dans laquelle elle apparaît : en cela, elle est toujours une rencontre entre une culture à un moment donné et le désir singulier d'un sujet nourri par le terreau de cette culture ; le phénomène est classique dans l'enseignement : les professeurs font (souvent silencieusement) le tri entre ce qui doit être retenu et oublié (c'est d'ailleurs ainsi que se crée la mémoire didactique de la classe– Brousseau, 1998). Cet espace de création a donc un prix pour le professeur : celui de la nouveauté sans prix ou d'une nouveauté marginale, d'un possible surgissement de ce que J. Giroux (2008) appelle les « conduites atypiques » [3].

La création mathématique exige donc du jeu c'est-à-dire un espace de liberté limité à la manière du jeu des gonds : sans jeu la porte ne peut s'ouvrir, trop de jeu la porte ne ferme plus.

2. Connaissance et savoirs : Les mathématiques toutes faites sont des mathématiques mortes

Saviez-vous que les corbeaux sont des êtres prodigieusement intelligents ! Certains même, les appellent les hominidés à plume. Ils sont capables par exemple de dénombrer des collections jusqu'à 17 ! Comment le sait-on ? Un jour deux éthologues voulaient étudier leur comportement dans leur milieu naturel, ils se donc sont installés dans une cabane avec leur matériel d'observation. Evidemment, les corbeaux se sont enfuis et nos éthologues ont attendus leur retour. Les corbeaux ne revenant pas, l'un des deux éthologues dit à l'autre : « Je vais sortir, ainsi ils reviendront ». C'est ce qu'il fit mais les corbeaux ne revenaient toujours pas. Alors l'autre sortit à son tour et quelques minutes après les corbeaux étaient de retour. Etonnés, ils recommencèrent l'expérience avec 3 personnes puis avec 4 et 5 éthologues ... les corbeaux ne revenaient toujours pas dès que le dernier n'était pas sorti ! Les corbeaux commencèrent à se tromper à partir de 18 !

Si j'avais plus de temps, je pourrais aussi vous raconter comment les corbeaux sont capables de résoudre des problèmes plus complexes : comment ils sont capables d'anticiper, de planifier des actions ... bref, les corbeaux réalisent des apprentissages extrêmement complexes dans des situations extrêmement variées ! Mais alors comment se fait-il que les corbeaux n'aient pas progressé dans leurs connaissances comme les hommes l'ont fait, alors même qu'ils sont capables d'une extraordinaire plasticité cognitive ?

La réponse est certainement très complexe mais je pense avoir au

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moins un élément de réponse : contrairement aux hommes les animaux ne capitalisent pas ce qu'ils ont acquis : il peuvent transmettre par ostension, par imitation ... bref, ils transmettent toujours dans une co-présence, dans une interaction immédiate (ie sans médiation) et ne gardent jamais une mémoire de ce qu'ils ont acquis pour la transmettre : c'est cette mémoire collective et collectivement constituée que nous appellerons « savoirs » ; les savoirs sont des formes instituées (au moins collectives) qui permettent de désigner (et de mémoriser donc) des manières d'agir, d'être, de faire, de sentir, de percevoir ... mais ce ne sont que des coquilles vides, des formes inertes, sans vie, des promesses d'action : e n'ai que le verbe « aimer » pour dire que « j'aime ma femme, ma fille, ma mère ... le poulet-frite, le hockey sur glace, Bach ou les mathématiques » et l'expression « Je t'aime » est bien vide de sens si elle n'est pas assortie de tout un ensemble complexe d'autres expériences (des regards, des signes empathie ...

Bref, sans les savoirs, pas de transmission différée (car il serait impossible de savoir quoi transmettre puisque il serait impossible de le désigner) ; il serait tout aussi impossible de désigner, de représenter nos expériences(singulières ou collectives) : les représenter pour les évoquer, les partager et donc les transmettre. Ce que je veux dire, c'est que les savoirs ne sont qu'une des conditions préalables de la transmission mais ne peuvent pas constituer pas l'objetde la transmission : car ce qui est visé par la transmission, par l'enseignement, c'est la re-production de l'expérience que permet de désigner ce savoir. Pour l'élève il ne s'agit de refaire encore (sur le modèle du professeur) mais bien de faire de nouveau. Par exemple, lorsqu'un professeur enseigne à ses élèves la manière de dériver des fonctions, il n'attend pas seulement qu'ils sachent calculer la dérivée d'une fonction particulière (ce n'est ici que la partie visible de l'enseignement et de l'apprentissage : le savoir), il attend surtout que ses élèves utilisent la dérivée pour traiter et résoudre des problèmes nouveaux c'est-à-dire des problèmes qu'ils n'ont jamais rencontrés : par exemple, pour un périmètre donné d'un rectangle, quelles doivent être les mesures de la longueur et la largueur pour obtenir la surface maximale ?

On voit bien à travers cet exemple l'intérêt de la distinction fondamentale qui est faite dans la théorie des situations entre « connaissances » et « savoirs ». Le professeur cherche à s'assurer à travers ce problème d'optimisation non seulement que l'élève sait sa leçon mais qu'il la comprise : que désormais il possède une bonne connaissance de la dérivation.

C'est entre savoir et connaissance dans cet espace articulant la culture, l'institution, la société (autrement dit tout ce qui est externe au sujet et auquel le sujet doit s'assujettir) et la singularité de nos expériences, de nos connaissances que je situerai le lieu de la *transgression*.

Explicitons un peu cette idée. Elle peut être formulée simplement : les mathématiques achevées c'est-à-dire telles celles que nous connaissons (les savoirs, les démonstrations, les algorithmes ...) sont des mathématiques mortes ; une grande partie du travail des professeurs consiste à créer les conditions de leur « résurrection » pour les élèves. C'est ici que l'essentiel va se jouer ! Pour ce faire, les professeurs n'ont pas d'autres choix que celui de créer des situations qui permettront à leurs élèves de rencontrer ce qu'ils ignorent et que pourtant ils doivent apprendre. Les professeurs espèrent que ces situations permettront aux élèves d'éprouver l'intérêt des connaissancesque le professeur cherche à leur communiquer : leurs usages, leurs fonctions, l'économie qu'elles rendent possible ...

Mais les professeurs ne peuvent pas se substituer à leurs élèves pour les leur apprendre (tout comme on ne peut marcher, parler, dormir, aimer ... ou souffrir à la place d'autrui, même si bien sûr, on met tout en œuvre pour l'aider ou le soulager dans ses apprentissage : c'est cette impossibilité qui constitue le noyau dur du *contrat didactique* (Brousseau, 1998 ; Sarrazy, 1995).

3. Le contrat didactique : de quoi s'agit-il ?

Toute relation d'enseignement peut être décrite sous la forme d'un contrat : je t'enseigne c'est-à-dire je te montre ou je te dis ... ce que tu dois apprendre et comment tu dois le faire et toi tu apprends c'est-à-dire tu reproduis ce que je t'ai dit ou montré. Il va de soi que ce contrat est implicite et n'a jamais été explicitement passé entre le professeur et ses élèves mais les uns comme les autres agissent comme s'il avait été conclu(de la même façon que lorsque nous parlons, nous faisons comme si les mots avaient le même sens ... pourtant rien n'est moins sûr.)

Mais ce contrat n'est pas tenable au moins pour deux raisons :

1. La première est que le professeur ne peut pas montrer ou dire à l'élève ce qu'il doit apprendrepuisque précisément l'élève ignore ce que le professeur cherche à lui enseigner ; s'il le lui montre alors l'élève est incapable de le voir et s'il le voit alors il est inutile de le lui montrer et donc de l'enseigner. Tel est un des premiers paradoxes que contrat : si l'élève comprend son professeur alors l'élève sait déjà mais alors il n'a pas besoin d'apprendre ; s'il ignore ce que le professeur cherche à lui enseigner alors il ne peut pas comprendre ce que son professeur lui dit !
2. La seconde raison tient à la nature même de ce que l'élève doit apprendre ; je l'ai déjà présentéeà propos de l'exemple de la dérivation des fonctions : ce n'est pas un *savoir* que l'élève doit ap-

prendre mais une *connaissance*, un usage, une manière de traiter un problème, une manière de faire des mathématiques … C'est cette même idée qu'exprimait Thurston à propos des demandes que lui faisaient les mathématiciens qui venaient le trouver pour obtenir quelques éclaircissements sur tel ou tel aspect d'une démonstration qu'il avait établie :« Ce que les mathématiciens avaient besoin, et ce qu'ils me demandaient, c'était d'apprendre mes façons de penser, et non comment je démontrais la conjecture dans le cas des variétés de Haken ».

J'illustrerai cette impossibilité du professeur à montrer ce qu'il veut par l'exemple que donne Wittgenstein (1961, XI, 325) de la double croix :

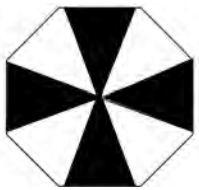

Figure 2.

Cette figure peut être vue comme une croix banche sur fond noir et comme une croix noire sur fond blanc) et pourtant la figure n'a changé : c'est ce « voir-ceci-comme-cela » que Wittgenstein appelle la naissance d'un aspect (la dérivée et son usage) (1985, §431). Cette impression visuelle peut être 'forcée' par l'imagination (par exemple, voir un animal dans un nuage), par la volonté (comme dans le cas de la double croix), par le savoir (voir un triangle en considérant un segment particulier comme étant sa base, puis en considérant un autre segment) … mais dans tous les cas, elle ne peut être montrée puisque le perçu étant resté invariable (*id.*, §440). C'est précisément sur ce point que Wittgenstein fait apparaître une confusion liée à l'amalgame de ces deux usages : décrire la vision de l'aspect (tantôt une croix noire, tantôt une croix blanche) *dans les mêmes termes* que la vision d'un état (« Je vois un chat »).

Tel est un des arguments des plus convaincants pour montrer que toute relation d'enseignement est fondamentalement déterminée à la fois par ce désir de tout dire du professeur et en même temps par cette impossibilité d'expliciter le sens (c'est-à-dire l'usage) : le contrat ne peut donc être tenu que s'il est rompu, violé, bref transgressé … car il exige de l'élève une *création* singulière correspondant à un usage nouveau mais conforme aux règles des mathématiciens.

Mais vous pourriez m'opposer la contradiction suivante : comment est-il possible à la fois de dire que l'élève réalise une création, qu'il agit conformément à la règle, tout en convoquant l'idée de transgression. Vous auriez raison à condition toutefois que je considère que l'action de l'élève est déterminée par la règle, que l'élève agit en fonction de la règle : or, il n'en est rien ! Car l'idée selon laquelle l'élève interprèterait la règle pour agir n'est pas correcte ; elle contient un paradoxe que Wittgenstein formule ainsi :

> « Aucune manière d'agir ne pourrait être déterminée par une règle, puisque chaque manière d'agir pourrait toujours se conformer à la règle » (1961, §201).Comment alors une règle peut-elle nous *guider* « puisque nous pouvons interpréter son expression de telle et telle autre façon ? » (1983, 282). S'il est possible de la suivre comme on veut, il est alors impossible de la suivre puisqu'elle ne nous contraint pas !

Tel est d'ailleurs le drame de ces élèves qui 'savent leur leçon mais qui ne comprennent pas'. Doit-on déplacer le problème en lui enseignant des méta-règles comme le suggèrent régulièrement certains auteurs ? Pas plus car non seulement l'élève ne fera plus de mathématiques mais on retrouverait à un méta-niveau les mêmes difficultés que l'enseignement des méta-règles était pourtant censé réduire !

Le paradoxe n'est qu'apparent et permet de dévoiler la faiblesse d'une conception mentaliste des rapports entre règle et action. Comme l'explique Bouveresse, ce paradoxe provient de la tendance que nous avons « à instaurer artificiellement entre la règle et ses applications une distance problématique qui, en réalité, n'existe pas » et qu'un hypothétique mécanisme mental permettrait de réduire (Bouveresse, 1986) car la règle ne contient pas en elle-même ses conditions d'application car « c'est précisément parce [qu'elle] doit pouvoir me servir à chaque instant de norme pour juger ma performance qu'elle ne peut pas me faire faire ce que je fais de la manière dont le ferait un mécanisme quelconque » (Bouveresse, 1986, 30).

Il est donc fondamental de ne pas amalgamer : « Suivre une règle » et « obéir à la règle ». « Suivre une régle » est une « création normative » selon l'expression de Hacker et Backer permettant d'estimer la conformité de l'action à ce que dit la règle [4]. Sa signification *correspond* à l'usage circonstancié que le sujet en fait *hic et nunc.*

4. C'est en cela qu'apprendre à suivre une règle est analogue à l'apprentissage d'un langage dans lequel « la grammaire nous autorise à faire certaines choses avec le langage et non certaines autres ; elle détermine le degré de liberté [. . .] les règles sont fixées et données : elles autorisent certaines combinaisons et en interdisent d'autres. » (Wittgenstein, 1988, 8 ; 94).

> « Une ligne ne me contraint-elle pas à la suivre ? – Non, mais quand je me suis décidé à l'utiliser *ainsi* comme modèle, elle me contraint. Non, c'est *moi* qui me contrains à l'utiliser ainsi. » (Wittgenstein, 1983, 329).

C'est dans cette décision que se situe la dimension créative (car non contenue dans la règle, dans la norme, dans la loi . . .) et donc transgressive !

L'apprentissage se manifestera lorsque l'élève se montera capable non de reproduire ce qu'on lui a dit ou montré, mais bien lorsqu'il produira de lui-même une conduite nouvelle (« nouvelle » car se manifestant dans une situation nouvelle) mais conforme à ce qui lui aura été enseigné. C'est en cela que si la transgression brave, déborde, viole, dépasse, franchit . . . les normes, les règles, les interdits, elle ne les détruit pas. Au pire, la transgression peut conduire à les interroger, à révéler leur obsolescence, leur caractère arbitraire. J'y reviendrai.

L'élève ne peut apprendre que s'il accepte de ne plus être enseigné en s'engageant de lui-même dans une activité par laquelle il pourra apprendre ce que le professeur ne peut effectivement lui montrer (un usage et donc un sens). Mais cette rupture ne peut se réaliser que sous certaines conditions ; ainsi, dans cette perspective, « enseigner » consistera à créer les conditions du surgissement de cette re-production par l'élève, non au sens de la copie, de la répétition de ce que le professeur a dit ou montré, mais bien au sens d'une production nouvelle dans une situation nouvelle. Cet espace de création a donc un prix pour le professeur : celui d'un possible surgissement de ce que J.Giroux (2008) appelle les « conduites atypiques » [5]. Tel est selon moi un des arguments principaux à l'encontre d'une lecture behavioriste de la théorie des situations didactiques.

Donnons un exemple de ce type de situationqui permet de restaurer une véritable activité mathématique : chercher, conjecturer, confronter les convictions, convaincre, prouver . . . bref *faire* l'expérience de *faire* des mathématiques – un *faire faire* dirait Conne (1999).

Un exemple extrait de GuyBrousseau (1998)

Situation d'étude des applications linéaires : l'agrandissement du puzzle.

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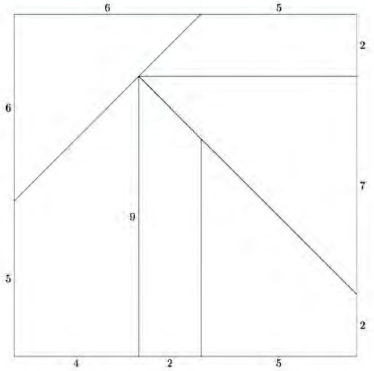
5. Elles ontcomme caractéristiques : « 1. Caractère marginal ; 2. Non adaptée aux contraintes ; 3. Spécifique à l'enjeu de la situation mathématique », c'est d'ailleurs en cela qu'on peut les considérer comme relevant du domaine didactique. Elles ne sont pas assimilables à une « conduite inefficace ou déviante » ou « inadaptée au problème proposé », et comme le précise, J.Giroux, elles « témoignent du rôle de la dimension antagoniste du milieu [de la situation] » et « sont donc corollaires de l'appropriation d'un véritable enjeu de la situation. »



Consigne
« Voici des puzzles. Vous allez en fabriquer de semblables, plus grands que les modèles, en respectant la règle suivante : le segment qui mesure 4 centimètres sur le modèle devra mesurer 7 centimètres sur votre reproduction. Je donne un puzzle par équipe de 5 ou 6, mais chaque élève fait au moins une pièce. Lorsque vous aurez fini, vous devez pouvoir reconstituer les mêmes figures qu'avec le modèle ».

Déroulement
« Après une brève concertation par équipes, les élèves se séparent. Le maître a affiché au tableau une représentation agrandie des puzzles complets. Presque tous les enfants pensent qu'il faut ajouter 3 centimètres à toutes les dimensions. Le résultat évidemment, c'est que les morceaux ne se raccordent pas. »

Dans cette situation, on voit clairement comment les élèves peuvent apprendre par interaction avec la situation ; elle permet (entre autres choses) d'invalider le modèle prégnant de l'addition : si $4 \to 7$ $(4+3)$ alors $5 \to 8$ $(5+3)$. Si on s'était limité à fournir un puzzle à chaque élève pour illustrer l'usage de la proportionnalité, les élèves auraient-ils appris les mêmes mathématiques ? Certainement ! Les élèves auraient appris les mêmes fonctions mais avec la première situation, ils apprennent aussi *quelque chose de plus* : une manière de faire des mathématiques. Autrement dit, ce qui diffère entre ces deux situations ce n'est pas le savoir mathématique mais bien la nature même de *l'expérience mathématique* qu'ils auront ici vécue. La première permet de conserver la nature même de l'expérience mathématique (recherche, conjecture ...), la seconde (basée sur l'ostension) montre le même savoir, mais ne permet pas à l'élève de faire l'expérience de cette manière de faire des mathématiques. On voit bien aussi à travers cet exemple non seulement le caractère indicible du contrat, mais aussi l'intérêt de rompre avec l'opposition classique entre les pédagogies dites « actives » et « classiques » qui constitue, me semble-t-il, l'une des impasses contemporaines des plus tenaces et contre-productives, car la connaissance des algorithmes ne détermine pas plus la connaissance de l'arithmétique que celle des règles du jeu d'échec ne détermine le « savoir joueraux échecs ». L'idée est simple mais pas triviale.

4. Transgression et sensibilité au contrat

La *sensibilité au contrat* est un concept que nous avons introduit pour désigner les décisions des élèves à l'égard des implicats mobilisés au sein du contrat didactique[6].

Donnons un premier exemple :

> La scène se déroule dans une classe de CM1. Quelques jours avant cet épisode, le professeur avait enseigné un algorithme permettant de calculer rapidement la différence entre deux nombres :
>
> $$\begin{array}{r} 328 \\ -\ 47 \\ \hline 281 \end{array} \ \xrightarrow[+3]{+3}\ \begin{array}{r} 331 \\ -\ 50 \\ \hline 281 \end{array} \ \xrightarrow[+50]{+50}\ \begin{array}{r} 381 \\ -100 \\ \hline 281 \end{array}$$
>
> Dans la première partie du contrôle semestriel, elle avait inclus l'exercice suivant :
>
> Quel serait ton cheminement pour effectuer ces calculs ?
>
> a) $875 - 379 =$ ________________
> b) $964 - 853 =$ ________________
> c) $999 - 111 =$ ________________
>
> Sur, 19 élèves 16 appliquent la règle enseignée pour le 3^{me} exercice :
>
> $$999 - 111 = 1008 - 120 = 1088 - 200 = 888$$

L'effet capitaine n'est rien d'autre qu'un simple et habituel effet de contrat de la même nature que celui-ci et certainement beaucoup plus spectaculaire. Dans ce type de situation, le professeur ne peut pas dire

6. Rappelons que le *contrat didactique* est défini par G. Brousseau comme étant « l'ensemble des comportements (spécifiques [des connaissances enseignées]) du maître qui sont attendus de l'élève et l'ensemble des comportements de l'élève qui sont attendus du maître. » (1980, 127). On peut aussi se reporter à la note de synthèse parue sur ce concept (Sarrazy, 1995) dans laquelle nous faisons apparaître les raisons de sa genèse, de son évolution dans le champ mȩme de la didactique et les usages qui en sont faits dans diverses communautés scientifiques.

ce qu'il attend des élèves (comme pour tout autre problème du reste), et on imagine que ceux-ci peuvent s'interroger sur la teneur de ses attentes : doivent-ils manifester, comme ils le font habituellement, leurs compétences en arithmétique en résolvant un problème dont tout porte à penser qu'il relève de l'addition, et ignorer (ou feindre d'ignorer) l'aspect quelque peu cocasse de la question ? Ou bien doivent-ils se prononcer sur la pertinence de la question en regard des informations contenues dans l'énoncé ?

Selon la « réponse » qu'ils donneront à ces « interrogations », soit ils répondront, comme le fait la majorité des élèves, « 36 ans », soit ils rejetteront la validité de ce problème en déclarant qu'ils ne peuvent y apporter une réponse raisonnable. C'est précisément ce positionnement à l'égard de cet implicat, que nous désignons « sensibilité au contrat didactique ».

Pour finir de préciser le sens de ce concept, examinons les deux extraits d'entretiens suivants :

Lou (10 ans), excellents résultats scolaires :
une logique de la transgression

– A ton avis, comment le professeur voit-il qu'un élève a compris une leçon ?
– (Lou) : il le voit en posant des questions qui sont un peu à côté de ce qu'il avait dit ; si l'enfant répond comme il faut, c'est qu'il a bien compris. Mais dans les évaluations, il faut répondre qu'un *petit détail* ; il y a des enfants qui répondent tout parce qu'ils ont appris bêtement sans rien comprendre ; ils ne sont pas capables de répondre qu'un détail. Si on met exactement le détail qu'elle voulait alors la maîtresse voit qu'on a bien compris.

Par contraste avec celui de Jean, on mesure combien les élèves ne sont pas tous également préparés à identifier et à décoder ces attentes implicites :

Jean (10 ans) : une logique de la répétition
faible en mathématique (bon en langue)

Le professeur voit qu'on a compris quand on écrit beaucoup et quand on écrit vite. Quand il donne des devoirs à la maison certains élèves se font aider par leurs parents alors à l'école il nous met tout seul. Comme ça, il est sûr qu'on ne copie pas et elle voit si on a compris. A la maison j'apprends des choses qu'il nous a dit, mais *j'ai remarqué qu'il ne nous demande pas vraiment ce qu'elle nous a appris alors moi je n'apprends pas vraiment ce qu'il nous a donné*car il change des choses.

Raisons didactiques pour l'une, qui, manifestement, a compris que l'apprentissage n'était pas répétition et que sa maîtresse était forcément tenue au silence pour des raisons profondément didactiques (« Elle a besoin de voir » dit-elle) ; raisons instrumentales pour l'autre, qui, tout en comprenant aussi la nécessité qu'a la maîtresse de regarder « comment on est », ne comprend pas vraiment pourquoi « elle ne refait pas vraiment ce qu'elle [nous] a appris » et qui, en toute logique, « n'apprend pas vraiment ce qu'elle a donné ».

Ces deux exemples suffisent à bien montrer comment s'établit un commerce inégal des implicites dans les rapports entre professeur et élèves, et en quoi l'absence d'analyses de ses conditions de production, contribue à entretenir l'idéologie charismatique à propos des succès et insuccès scolaires (et probablement de façon plus marquée en mathématiques). C'est à l'analyse des conditions de production des sensibilités au contrat didactique que nous nous attacherons maintenant.

4.1. L'effet des styles pédagogiques sur les phénomènes de sensibilité au contrat ?

L'analyse des phénomènes de sensibilité au contrat est complexe ; elle se situe au croisement des deux principaux univers de pratiques des élèves : l'école et la famille. Nous n'aborderons ici que l'analyse du champ scolaire [7].

Pour comprendre les raisons de l'inégale dispersion de la sensibilité selon les classes, le modèle d'analyse devait permettre d'estimer la marge de manœuvre qui, dans l'organisation et la gestion des situations, est dévolue (volontairement ou non) aux élèves.

Résultats et commentaires

Nous nous limiterons ici à la présentation des deux styles les plus contrastés :

1. Le style « *actif* » correspond à ce qu'on pourrait appeler en première approximation une 'pédagogie active'. Il se caractérise par une forte variabilité dans l'organisation et la gestion des situations : ces maîtres pratiquent régulièrement le travail par groupes sans se limiter forcément à cette forme de groupement des élèves ; les problèmes 'amorces' sont généralement complexes ; leur classe est fortement interactive (les élèves interviennent spontanément, les réponses 'chorales' ne sont pas rares ...) ; l'institutionnalisation est différée dans la leçon. Telles sont les traits principaux de ce premier style.

7. Le lecteur pourra trouver dans Sarrazy (2002) l'étude relative à l'impact des pratiques d'éducation familiale sur les phénomčnes de sensibilité au contrat didactique.

2. Le style « *académique* » se caractérise par une faible ouverture et une faible variété des situations ; on pourrait l'appeler 'enseignement classique' ou 'frontal' dont le schéma de base pourrait se résumer par le triptyque « montrer-retenir-appliquer ». Ces maîtres institutionnalisent un modèle de résolution très rapidement puis soumettent à leurs élèves des exercices de complexité croissante ; ils sont d'abord corrigés localement (le maître passe dans les rangs et corrige au 'couppar coup') puis collectivement, au tableau, où il enseigne en commentant la solution, s'aidant parfois, selon le temps dont il dispose, de la participation de certains élèves sur le mode « question-réponse ». L'espace interactif est quantitativement et qualitativement fort différent de celui du style précédent : on n'observe quasiment jamais d'interventions spontanées ou de réponses 'chorales' des élèves. Bref ce sont des maîtres très formalistes qui cherchent à maîtriser le plus de paramètres possible de leur classe.

Dans des contextes 'actifs', 48% des élèves s'autorisent à produire une réponse sans calculer (au problème escargot) contre seulement 17% dans le contexte 'académique' [$\chi^2 = 6.08$; $p. < .04$]– bien entendu, ces différences se maintiennent à même niveau scolaire et quel que soit le type de situation de production de ces réponses[8]. Ces styles s'avèrent donc pertinents pour expliquer les phénomènes de sensibilité au contrat comme le montre le graphique ci-dessous :

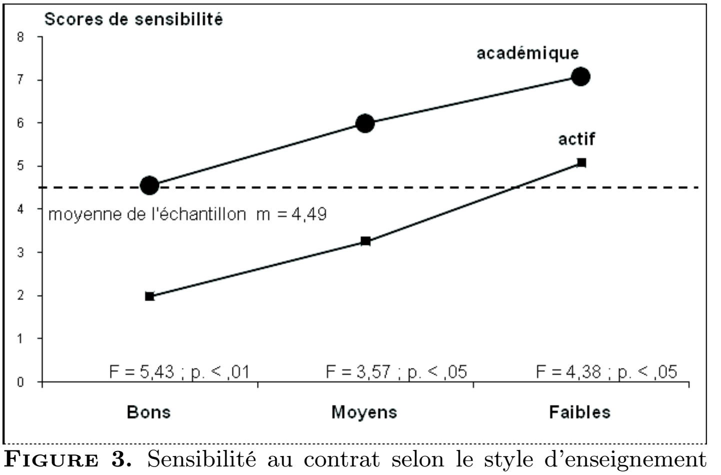


Figure 3. Sensibilité au contrat selon le style d'enseignement pour chacun des niveaux scolaires en mathématiques.

8. Notons aussi que les mȩmes résultats ont été enregistrés avec les autres types de problèmes : problèmes type 'capitaine', lacunaires, etc.

Les différences entre les cultures didactiques, descriptibles par les formes d'organisation et de gestion des situations, permettent d'expliquer l'inégale distribution des sensibilités des élèves. Plus les élèves ont la possibilité de confronter les règles enseignées à des situations faiblement ritualisées, comme c'est le cas dans les contextes actifs, plus ils 's'autorisent' à les engager dans des situations nouvelles. Réciproquement, plus l'incertitude attachée aux situations est réduite, comme c'est le cas dans les contextes académiques, plus les élèves semblent établir un rapport 'rigide' entre une règle et son usage et ne 's'autorisent' pas, ou très peu, des écarts non conventionnels.

5. Efficacité et équité : cas de l'arithmétique

Peut-on dire qu'un style serait préférable à un autre, arguant que le style « actif » permettrait aux élèves de « mettre plus de 'sens' sur les savoirs scolaires » pour reprendre ici le credo pédagogique actuel ? Ce serait une erreur.

Si nous proposons à ces mêmes élèves des problèmes de difficulté non triviale dans des situations faiblement décontextualisées par rapport au contexte d'acquisition, alors les résultats précédents s'inversent.

Conditions de l'expérience

Les problèmes retenus correspondent à la 4$^{\text{ème}}$ structure additive de la typologie élaborée par Vergnaud (1983). Cette structure présente la particularité de ne mettre en jeu que des transformations positives ou négatives ('gagner' ou 'perdre') sans qu'aucune indication ne soit fournie sur l'état numérique initial (d'où son appellation : « TTT », « transformation-transformation-transformation »). Exemple :

> Lou joue deux parties de billes. Elle joue une partie. A la seconde partie, elle perd 4billes. Après les deux parties, elle a gagné 6 billes. Que s'est-il passé à la 1$^{\text{ère}}$ partie ?

Le plan expérimental est classique : 22 problèmes de difficulté variable ont été soumis aux élèves lors d'un pré-test. Il fut suivi de 2 leçons, espacées entre elles d'une semaine, à l'issue desquelles les mêmes problèmes ont de nouveau été soumis aux élèves.

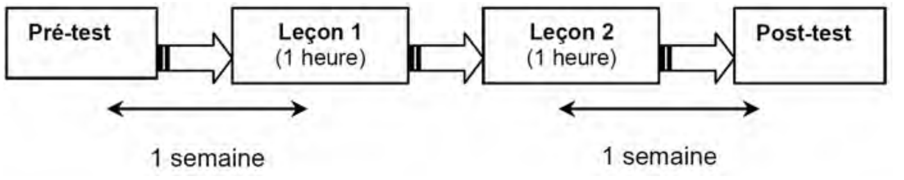


FIGURE 4.

Afin de ne pas influencer les scénarii des leçons, les professeurs n'ont eu accès au protocole d'évaluation qu'au moment du post-test. Un 'indice de progression' (I_p), qui bien sûr ne se réduit pas au simple calcul de la différence des scores du pré-test et du post-test, a été défini pour chacun des élèves [9].

5.1. Résultats et commentaires

Deux aspects ont été pris en compte pour évaluer les effets de ces enseignements :

1. Le premier, appelé *efficacité*, correspond à la mesure des performances effectives enregistrées au post-test en contrôlant les variables susceptibles d'infléchir les résultats observés (ici le niveau scolaire des élèves).
2. Le second, appelé l'*équité*, mesure l'efficacité différentielle pour un groupe d'élèves donné, en tenant compte, cette fois, de leur niveau initial (en l'occurrence, les résultats obtenus au pré-test).

Les deux graphiques ci-après résument les principaux résultats.

Le style 'académique' s'avère non seulement plus équitable(les élèves progressent significativement plus que les élèves 'actifs' – $F_1 = 3,73$; $p. < .05$), mais aussi plus efficace : leurs performances sont significativement supérieures ($F_1 = 5,10$; $p. < .01$). Ces effets sont particulièrement manifestes pour les élèves faibles (*Efficacité* : $F = 20,26$; $p. < .01$ – *Equité* : $F = 8,65$; $p < .01$).

Ces derniers résultats devraient-ils nous inciter à renverser notre précédente conclusion et affirmer, cette fois, qu'une 'pédagogie classique' est préférable à une 'pédagogie active' ? Conclure que celles-ci sont élitaires ?

9. Le modèle d'estimation des progrès utilisé ici est d'une construction complexe ; la procédure utilisée (construction d'un modèle théorique) d'une part permet d'éviter les effets classiques de plafond ou de plancher et, d'autre part, autorise à affirmer que l'élève a progressé (régressé) au seuil de risque de 10%.

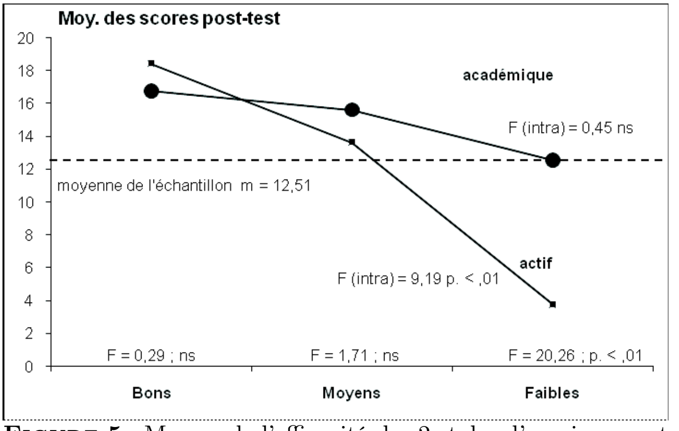


Figure 5. Mesure de l'efficacité des 2 styles d'enseignement selon le niveau en mathématiques des élèves.

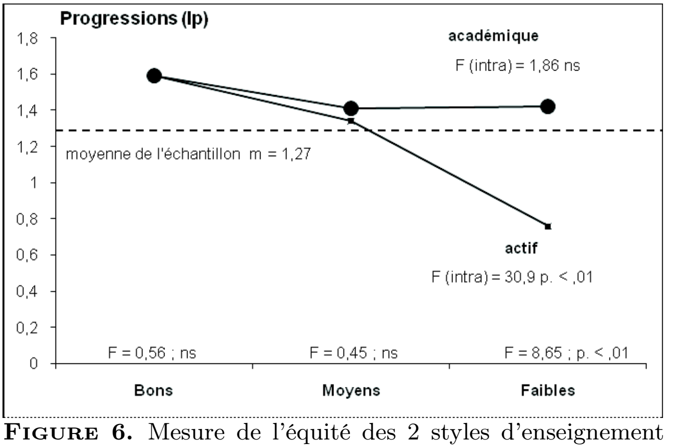


Figure 6. Mesure de l'équité des 2 styles d'enseignement selon le niveau en mathématiques des élèves.

Ce serait très largement imprudent. Car, d'une part, toute recherche n'est qu'un instantané, une fenêtre ouverte sur un univers de pratiques dont les temporalités ne sont pas analogues (cf. la thèse de Chopin, 2007). Rien ici, ne permet d'affirmer que sur un nombre plus important de leçons – qui, je le rappelle, ont été limitées, en accord avec les professeurs, à deux – les performances auraient été les mêmes. Le tempo de l'apprentissage est probablement plus lent dans un style actif (mais aussi le temps alloué pour l'enseignement a des effets très significatifs sur la structuration et la gestion des dispositifs didactiques, Chopin, *id.*, 2007). Enfin, les résultats précédemment obtenus à propos de la sensibilité ne peuvent pas être ignorés.

6. Conclusion

L'ensemble de ces résultats appelle la question fondamentale de la visée de l'enseignement. Une 'tête bien pleine' ou une 'tête bien faite' ? Un 'bon' enseignement doit-il viser une bonne maîtrise des algorithmes ou bien permettre aux élèves de les utiliser dans des situations nouvelles ? Cette question n'est pas seulement scientifique, elle est aussi foncièrement et noblement politique car elle pose inévitablement celle de savoir quel type d'hommes et de femmes l'Ecole doit former. Or manifestement, si les deux visées précédemment énoncées paraissent nécessaires ensemble, elles apparaissent, de fait, peu compatibles par les rapports paradoxaux qu'elles entretiennent. C'est là le « paradoxe de la sensibilité », que j'avais énoncé en 1996 : « Plus le maître cherche à enseigner clairement l'usage d'une règle, plus il réduit la possibilité d'un usage singulier, mais idoine et qu'il exigera pourtant ultérieurement. ». On comprend mieux maintenant comment peut se tisser le drame didactique qui se joue pour les élèves les plus faibles : aveuglés par l'algorithme et par la certitude assurée par leur maître quant à son efficacité pour traiter une infinité de situations, ils ne s'autorisent pas à envisager d'autres usages que ceux qu'ils ont initialement rencontrés et, comme le disciple à qui son maître montre la lune, ils regardent son doigt.

Pour finir, et pour résumé on dira qu'apprendre des mathématiques ce n'est pas seulement mémoriser des mathématiques mais re-créer leurs usages ; et créer c'est s'autoriser à découvrir ce qui n'est pas, c'est aller au-delà *(trans ... gresser, trans ... cender)*. Cette autorisation met en tension dialectique obéissance et transgression car on ne saurait imaginer une possibilité de création sans transgression mais en même temps il ne saurait y avoir de transgression sans loi, ni normes mais aussi sans la libre acceptation des contraintes qui définissent l'espace de sa création (cela est très vrai en mathématique). On l'a vu ces espaces de liberté n'existent que parce qu'il y a eu transmission et les conditions de la transmission (la manière d'enseigner) déterminent les possibilités de transgression : telle est notre responsabilité d'éducateur !

Finissons sur un vœu : nos élèves devront réinventer le monde de demain, ils devront aller au-delà de ce que nous leur avons transmis ... notre travail, nous éducateurs, est de leur permettre de vivre à l'école cette expérience de la transgression, de la démesure. Comment espérer sinon qu'ils le fassent, si on ne leur a jamais donné l'occasion de la vivre.

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Looking at students' mathematics: from a deficit view on mathematical knowledge toward possibilities of mathematical actions

Abstract. Most work in mathematics education (research and teaching) focuses on students' learning of mathematics. The (tacit) orientation taken is to look at that learning and create a conjecture about what students know or do not know, and then try to find ways to help students develop, or better understand, this mathematical knowledge. In this paper, I suggest that there are two major difficulties with this attitude for mathematics education. The first concerns the fact that mathematical knowledge is seen as a thing, something that someone can grab, as if it existed independently by itself. This view on mathematical knowledge leads to a second difficulty, which is that it offers a "deficit" view of learning influenced by a medical paradigm. Considering mathematical knowledge as an external thing "to know about," unfortunately, leads to comparing students' mathematics with an allegedly external mathematics. With this orientation, students are always seen as lacking something, as needing more. They are always seen in deficit. I argue that this view is problematic for conceptualizing mathematical activity, and even ethically, and that a change is needed toward looking at what is made possible by students' actions and where it can lead, rather than focusing on something that is supposedly missing. This

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Key words and phrases: Mathematical learning, knowledge and actions, deficit view, mathematical possibilities.

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advocacy requires transformations of our current paradigms, and I present the theoretical groundings from various perspectives to support this view, as well as discussing the methodological shifts that this imposes for analyzing students' mathematics.

1. Context – prelude

> I knew very well how to teach fractions before starting my studies. [. . .] Now, I do not know anymore how to teach them and this is where the result of *didactique* of mathematics lies. I am not, in that sense, required to find the most adapted solution to my classroom from the many different possible options. Now that I know these possibilities, I feel obligated to change my conceptions within the student–teacher interaction, I have doubts, I see students' difficulties that I had never thought of before. It is the embarrassment of richness that is now the reason for my worries, for my doubts.
> (Krygowska, 1973; cited in Bednarz, 2000, p. 77, my translation)

I begin with this quote from the Polish didactician Krygowska in order to offer a context for this paper, specifically to offer a context of non-simplicity in relation to what are the outcomes of research in mathematics education and all the discourse about finding the "best ways" to teach or make students learn mathematics. Krygowska's words stress that research in mathematics education is not geared to finding optimal solutions, but to generating new ideas and understanding phenomena in depth; which clearly cannot lead to easy, one-size-fits-all predetermined practical solutions to solve predetermined problems; the famous "what works," a view that has been amply criticized in mathematics education (see Bednarz, 2000; Zaslavsky, Chapman & Leikin, 2003) and in education (see Biesta, 2007, 2010).

In this paper, I address ideas and issues that many others have addressed, explicitly or implicitly, through the papers in these proceedings and elsewhere. Thus I do not pretend to reinvent the wheel, but attempt to push it forward. The ideas and issues that I address here are a work in progress, part of an ambitious research program undertaken by my colleague Jean-François Maheux and myself and the students in our research lab [1], which aims at better defining the issues related to mathematical activity at an epistemological level.

Before engaging in these issues, other notes are in order to situate my work as researcher and better contextualize the ideas that I present here.

1. See http://www.epistemo.math.uqam.ca

As a researcher, I work in mental mathematics where I study solvers' various strategies and ways of solving various problems. I am thus deeply interested in mathematical strategies and activity and not in issues of knowledge of any sort. As well, my view of mathematics as a field of study is, along with what Jeppe Skott presented in the conference, aligned to Lampert's (1990) critique of how mathematics is portrayed in schools:

> Commonly, mathematics is associated with certainty: knowing it, with being able to get the right answer, quickly. These cultural assumptions are shaped by school experience, in which *doing* mathematics means following the rules laid down by the teacher; *knowing* mathematics means remembering and applying the correct rule when the teacher asks a question; and mathematical *truth is determined* when the answer is ratified by the teacher. Beliefs about how to do mathematics and what it means to know it in school are acquired through years of watching, listening, and practicing.
>
> (p. 32, emphasis in the original)

I also do not comply with this view of mathematics and prefer that proposed by Papert (n.d.):

> I like to say there is a big distinction between something that I love and I call mathematics and something called "math", which is what we teach in schools and that's not a mathematics curriculum it's a "math" curriculum. [...] Mathematics is an active intellectual activity, and it means working at things where you're using the mathematical ideas that you are struggling with for a larger purpose. And, the idea that the larger purpose could be discovering something that the teacher decided you got to discover is not a larger purpose.

This gives an important context to my work, and particularly the context of the ideas and issues I address here. I present these in the following pages, obviously constrained by the space allowed in these proceedings.

2. Mathematical knowledge and research

Most work in mathematics education research focuses on students' learning of mathematics. The usual (tacit) orientation adopted is to look at that learning and create a conjecture on what the students know or do not know, and then attempt to find ways of helping students to develop or better understand this mathematical knowledge.

I suggest that there are two main issues with this orientation for mathematics education research. Also, it is important to keep this in mind:

I raise these issues because they carry great potential to stimulate deep reflections, to trigger deep questions in our community of mathematics education researchers, and not because it represents a "wrong" path or point of view for mathematics education research.

2.1. Knowledge as a thing

The first issue worth reflecting on is that through this view, mathematical knowledge is conceived of as a thing, a commodity that someone can possess. Mathematical knowledge is seen as something that exists independently of the knowers themselves, as an external object that one must grasp. This creates a well-known problematic distance between students and mathematics, where teachers attempt to teach *that* external mathematics to students as best they can, and students try to learn *that* external mathematics. It participates in the alienation of knowers from mathematics itself, creating a notable distance between the two.

In our research lab, we are acutely uncomfortable with this situation, with this idea of mathematical knowledge seen as a thing that someone can possess. Some blunt questions are in order: Has anyone ever seen mathematical knowledge? If so, where? As Jeppe Skott in his presentation hinted, it is maybe not in people's heads, something that Kieren, Calvert, Reid and Simmt (1995) formulate this way: "Knowledge is not in the book or in the library; Knowledge is not in our heads; Knowledge is in the interaction" (p. 1).

We believe that what one sees when there is an assertion of "seeing" mathematical knowledge are mainly manifestations, enactments of, something that is done. As Bernard Sarrazy mentioned in his talk: "If I want to check if you know how to subtract, I ask you to do subtraction." This resonates well with Maturana's (1988) claim:

> Thus, if someone claims to know algebra – that is, to be an algebraist – we demand of him or her to perform in the domain of what we consider algebra to be, and if according to us she or he performs adequately in that domain, we accept the claim (pp. 4-5).

These manifestations, these enactments are something that, as observers, we assign as mathematical knowledge. But this reification process of these actions into knowledge, into things, need not be. In our research lab, we prefer to let go of this unneeded need to reify mathematical enactments into mathematical knowledge. My colleague and I focus rather on those actions that we observe as mathematical, what we have called in French *faire mathématique* (Maheux & Proulx, 2014a) or in English *doing/mathematics* (Maheux & Proulx, 2015). I return to these ideas below.

2.2. Deficit view

The second issue worth reflecting on is about the fact that this view of mathematical knowledge leads to what is often referred to as a *deficit* view of learning, influenced by a medical paradigm (see Bélanger, 1990-91). Considering mathematical knowledge as an external thing "to know about" unfortunately leads to comparing students' mathematics with this allegedly external mathematics. With this orientation, in these comparisons, students are always seen as lacking something, as needing more than this allegedly perfect knowledge base external to knowers. In a word, they are seen in deficit.

A number of sub-issues then emerge from these considerations. One, as mentioned above, is that it presents the idea that there is indeed some kind of mathematical thing to grasp, some predigested mathematical knowledge ready made to solve already predetermined problems. It reinforces a view of knowledge as static and predetermined. This leads also to an ethical concern, where students' mathematical creativity is constrained, because what students do/understand/produce is always seen through the lens of this comparison with this allegedly external mathematical knowledge base and thus is always a subset thereof. Students invariably then produce mathematics that is within mathematics that we already know and thus can never be mathematically creative. Borasi (1987) makes the bold assertion that this situation, where students' mathematical creativity is tamed, is representative of *our own* lack of creativity as researchers:

> It is interesting to note that an interpretation of errors solely as tools for diagnosis and remediation would have only partially exploited the educational potential of the error discussed. [...] In addition, the creativity of the researchers themselves when analyzing the error would be constrained by their limited focus on finding the causes of the students' error so that they could eliminate it. Thus they see the error necessarily as a deviation from an established body of knowledge, and do not even allow themselves to consider it as a possible challenge to the standard results (p. 4).

However, most important we believe, is that as researchers we see a problem with this. We see as problematic the fact that students lack knowledge, even if *it is we who are creating this lack* because *we* are those making this comparison, which evidently results in saying that students' lack knowledge. By making comparisons between students' mathematics and this allegedly external and perfect knowledge base, we are the ones creating this deficit between students' mathematics and this mathematical knowledge base. And, acting out of surprise, we scream in fear facing the situation; whereas this should hardly be a surprise since it is built into the comparison!

In our research lab, as mentioned above, our discomfort with this situation led us to focus on mathematical actions, on the doings of mathematics. I present some aspects of this view below, and then outline how and what this view on mathematical actions led us to develop in relation to data analysis.

3. Mathematical actions and possibilities, part 1: what is it?

One significant question to ask is "What do we mean by *doing|mathematics*?". It would take too much space to explain this in detail here, and we have already developed extensive arguments in three earlier publications (Maheux & Proulx, 2014a, 2014b, 2015). However, *doing|mathematics* goes beyond the usual views of producing, being the author of, developing, fabricating, elaborating, giving form to, constituting, accomplishing, making, being the cause of, determining a way of being, giving a quality, a character, or a state to something, changing, transforming, acting, behaving, and so forth. In a nutshell, *doing/mathematics* means a transformation related to the mathematical domain. And this obviously is continually in relation to the observer who claims/recognizes that transformation. Hence *doing|mathematics* is recognition *by an observer* of a transformation related to the mathematical domain *for that observer.*

Doing/mathematics is thus not a representation in relation to individuals (of personal knowledge) or to the discipline (of the established mathematical knowledge). *Doing/mathematics* as transformations are observable events that can participate in the elaboration of mathematical meaning. Thus remembering Maturana's (1988) quote about algebra, *doing|mathematics* is said to be mathematical if the observer gives it meaning and relates it to other *doing/mathematics* linked for that observer to the mathematical domain (defined by that observer). Going further by using Bateson's (1979, p. 228) concept of information, we can say that *doing|mathematics*is for an observer is an act of distinction of "any difference that makes a difference" at the mathematical level for that observer.

This focus on mathematical doings led us to propose a transformation of view, not one focused on a state of affairs ("it is" *versus* "it is not") but toward the possibilities, the potential of these actions, to where they lead. In that sense, we are aligned to the mathematician Dave Henderson's (1981) view of mathematical correctness:

> I relate correctness to the goal by saying that something is correct to the extent it moves an individual or group of individuals in the direction of an expanded understanding and perception of reality. [...] In particular, an argument is correct to the extent that it expands a person's understanding and perception. So what's correct depends both on reality and on the individual. I claim that this is what we all naturally try to do whenever we are involved in understanding or communicating mathematics. How do we view mathematical arguments? When do we call an argument good? When do we consider it convincing? – When we're convinced! – Right? – When the argument causes us to see something we hadn't seen before. We can follow a logical argument step by step and agree with each step but still not be satisfied. We want more. We want to perceive something (p. 13.)

This offers a view oriented toward what can be made possible by these mathematical actions, what it can lead to, where it extends; rather than a focus on what is supposedly missing, a state of affairs of what one does know or does not know. I discuss this orientation below.

4. Mathematical actions and possibilities, part 2: data analysis

As researchers, in our research lab, we are interested in the potential, in the possibilities, in the extensions, or even in the future one might say, of what students do, of their mathematical actions. This view requires an important methodological shift for analyzing students' mathematics (something we begin to discuss in Maheux & Proulx, 2015).

With this positioning, the stakes of analyzing data rest no longer in the truth or validity of students' mathematics, but in what they offer to oneself and another. Thus, the approach engaged in for data analysis implies the *necessity* to move away from questions about mathematical knowledge or knowing and focuses on students' actions *for imagining possibilities for mathematics education*, for seeing extensions, rather than arguing for or against taken-as-given practices, activities, tools, and so forth. Following Jarvis' (2004) idea of speculative thinking, the intention is to imagine possibilities, to draw them out by analyzing students' mathematics.

Even if it prevents us from making direct assumptions about what students might know or not know – as if they were holding knowledge one way or another – studying students' mathematical actions makes it possible to make sense of these propositions as diverse ways to *approach* and *go about* students' mathematical activity. That is, regardless of the "trustworthiness" of the students' oral account of their thinking processes, or the possible relation of the strategies observed with fixed/preexisting forms of knowing, these strategies can be discussed in terms of *action possibilities*. Inspired by Châtelet's (1987, 1997) notion of the virtual, analysis of the strategies is conducted in order to produce interpretations that *open* to a mathematical plan for creating possibilities out of those strategies; an orientation that does not aim to reify and map the mathematical activities engaged in (e.g. ways of reading an equation, of solving a problem), because this would tame them and fix them. In contrast, Châtelet suggests keeping them alive, seeing them as "creators of possibilities," as moving powers, as provocations. These possibilities of making sense of, of creating meaning about students' mathematical activity are central to our community of mathematics education researchers.

The language used in terms of the observer's point of view transforms assertions about what are seen as findings and what is learned from them: it points to the virtual, to potential extensions, it opens to what could be created by it and how it can make us think in different ways. Thus when we speak of mathematical potential, of its virtuality, of its future, it is always from the point of view of judgment of the observer, in what this observer considers mathematically potential. In order to illustrate these ideas about analysis in terms of potentialities, in the following section I analyze three examples taken from my own studies on mental mathematics.

5. Mathematical actions and possibilities, part 3: examples

5.1. Example #1: solving algebraic equations

In a study about mental algebraic equation solving, adult solvers (future teachers) were asked to solve various usual algebraic equations mentally (they were given 15-20 seconds to solve). These equations were of the form $Ax + B = C$, $Ax + B = Cx + D$, $Ax/B = C/D$, $Ax^2 + Bx + C = 0$ (see Proulx, 2013a, b, for more details).

As a first example for discussing the perspective on data analysis in terms of potentialities and extensions, Figure 1 illustrates one of the strategies engaged in to solve the equation $5x + 6 + 4x + 3 = -1 + 9x$. The participant explained that there was no solution, because we can easily see $9x$ on both sides of the equality, and that on each side, without adding them, there were remaining numbers that did not resulted in the same value. This leads to the conclusion that no x exists that can make different numbers become equal.

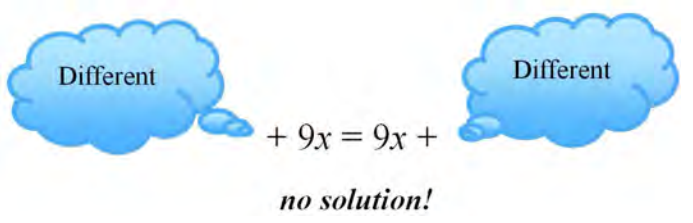

Figure 1. Example of the global reading strategy.

One finds in this a global reading strategy of the equation, which enables a rapid answer and puts forward the importance of analyzing the equation to solve without plunging into its mechanical resolution. In addition, this strategy permits the solving of a number of complex tasks that can be analyzed globally. For example, for $x + x^2 = 2x^2 + 5 - x^2$, a global reading can lead one to see that there are redundancies on each side of the equality sign, in x and in x^2, thus leading one not to consider them in the solving and asserting that $x = 5$. With this global strategy, the "noise" provoked by the exponent 2 in x^2 is avoided on the basis of repetition; in the same way as for the presence of a fraction in the well-known example $x + \frac{x}{4} = 6 + \frac{x}{4}$ from Bednarz and Janvier (1992). This global reading strategy, focused on the search for a value of x that renders the equality true, leads to an analysis of the equation to solve rather than entering into its solving "head first".

5.2. Example #2: operations on functions

In another experiment, Grade-11 high school students (15-16 years old), had to solve graphically usual tasks about operation on functions (see Proulx, 2015a). The graph of two (or three) functions were represented in the Cartesian plan on the whiteboard, and students had 15-20 seconds to operate on these functions and then draw their response on a blank sheet with a Cartesian graph on it (with the line $y = x$ drawn on it as a referent). Figure 2a illustrates a task where students had to add functions f and g, and Figure 2b displays one of the strategies developed to solve it. To solve the $f + g$ task, students paid attention to the following points: (1) where f cuts the x-axis (x-intercept), resulting in an image-length in g (because the image-length in f is 0); (2) where f and g cross each other and have the same image-length, resulting in an image double the value than where they intersect; (3) where f and g cut across the y-axis (y-intercept), resulting in a similar process as in (2); (4) where g cut the x-axis as in (1).

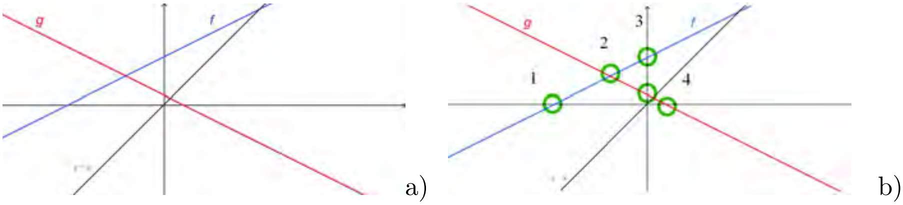

Figure 2. (a) The $f + g$ task; (b) The strategy focused on significant points to solve it.

This strategy, focused on significant points that enable determining where the line/function is, has potential, e.g., in relation to an extension toward multiplication of functions. Indeed, paying attention to points 1 and 4 permits one to evaluate the general shape of image-length for an x smaller than that at point 1: with negative values of f multiplied with positive images of g giving negative values in its multiplication; the same is true for images with an x bigger than the one of point 4. For values in x situated between points 1 and 4, the multiplication of images gives a positive value, leading one to see the quadratic function (2^{nd} degree) created by the multiplication of two linear functions (1^{st} degree). This entry through points of significance opens toward a sensitivity to graphical elements related to these functions, what can also be linked to the study of inflexion points and zeros (obtaining in this case the following analytical table $[+ \mid - \mid +]$).

5.3. Example #3: systems of linear equations

In another experiment, adult solvers (high school teachers) were given 15-20 seconds to solve a number of usual tasks on systems of equations, given algebraically on the board, and then to draw their response on a blank sheet with a Cartesian graph on it (also with the line $y = x$ drawn as a referent) (see Proulx, 2015b, for more details). In the case of solving the following system of equations "$y = x$ and $y = -x+2$, Figure 3 shows the answer given by one participant that is, the line $x = 1$.

The participant drew the vertical line, that is $x = 1$, explaining that he did not have enough time to find the value of y, but that the solution had to be on this line because when replacing $x = 1$ in each equation, it gave the same value. Of note is that the substitution of $x = 1$ in the equations directly gives the value in y (equations being of the form $y = mx + b$). However, in his algebraic manipulations to find the value for x, the emphasis is on finding a common x that gives the same answer ($x = ?$ and $-x + 2 = ?$) and not on finding the value for y even if *it is* the same value. But in his strategy, both were done/seen separately.

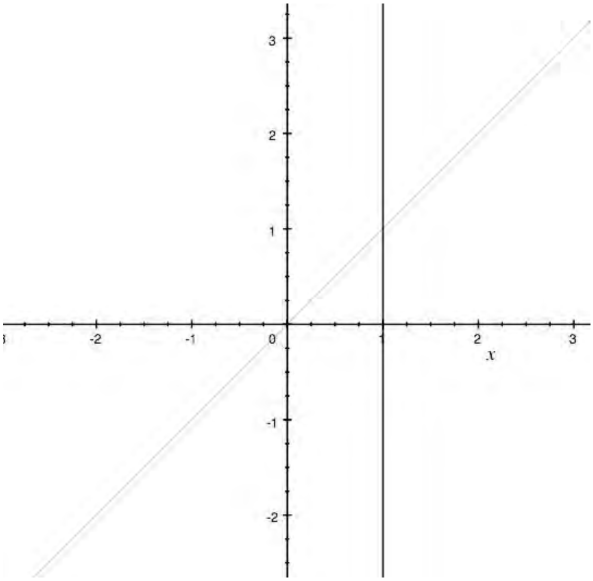

Figure 3. The line $x = 1$ given as the answer to the system "$y = x$ and $y = -x + 2$.

Even if incomplete, this strategy introduces many mathematical possibilities and extensions. The line $x = 1$ permits representation of all possibilities for y to solve the system, even if only one value will be able to satisfy both equations simultaneously. In addition, this line $x = 1$ represents a family of solutions to the system of parametrical equations "$y = x + k$ et $y = -x + (2 + k)$", leading to the study of parameters with $k = 0$ for the parameter of this system that has $(1, 1)$ as a solution. Also, this "omitting" to pay attention to y highlights an interesting obvious fact that the value for $x = 1$ for both equations is the value for y, and thus doing this is also working on the value in y because the value in y needs to satisfy both lines of the system of equations. Finally, the intersection of $x = 1$ and the referent line $y = x$ is exactly where the solution point is situated, permitting one to insist that the solution *is part of* both lines of the system; obvious facts often buried under the algebraic requests and that this strategy underlines.

6. Concluding remarks

Little more can be said, and it would not be interesting to restate everything here. Of importance, however, is the orientation that we take toward students' mathematical actions, not seen from a deficit point of view, or as things that one can take and grasp. This is an important point, because some might be tempted to think that we aim to oppose the deficit view with some kind of "positive" view, focusing on what students know and not on what they do not know (the metaphor of the glass half empty versus the glass half full). Our proposition toward mathematical actions and analyzing them in terms of potential is differnt. The deficit and the positive views represent both sides of the same coin. Both offer a consideration of a *state of affairs*, of what is, and of knowledge being a thing that someone has or does not have. As mentioned above, we are not comfortable in our research lab with any of these positioning, and are orienting ourselves toward the possible, the potential, the extensions, the future of mathematical actions; a "possible" that, we believe, has much potential for the future of mathematics education research.

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Affective transgression – a new perspective on the problem of low achievement in learning mathematics

Abstract. The idea of bringing the concept of *transgression* into the field of mathematics education has emerged from a reflective approach toward the practice of teaching. The considerable number of low achieving students, the "underachievement syndrome", and the phenomenon known as *math anxiety* are examples of problems for which the field of mathematics alone is insufficient to provide either prevention or remedy. Thus, mathematics educators take up the challenge of exploring the diversity of sciences to search for effective tools in other domains, for instance psychology, philosophy and pedagogy. An important goal is to test such tools and introduce the most promising findings to the research community. This paper offers a new perspective on the problem of low-achievement, drawn from study of the literature on affect in mathematics education and the author's adaptation of the psychological concept of transgression.

1. Introduction

Mathematical competencies are necessary to many domains of life and labour. At every stage results achieved in mathematics become the gateway to further education and realization of professional aspirations of the individual. Nevertheless, many students achieve unsatisfactory learning results. It is unsettling that each year, among those students in Poland who do not pass their high school final exam there are many who fail in mathematics. Gruszczyk-Kolczyńska and Zielińska (2009) highlighted the complexity of the problem:

Key words and phrases: affective transgression, emotions, mathematics related beliefs, low achievement in learning mathematics, math anxiety..

AMS (2000) Subject Classification: Primary 97C20, Secondary 97C50, 97D20.

> The situation of students who encounter failures in learning mathematics is so complicated, that it is hard to help them effectively. (...) Delving into the sources of this phenomenon might contribute to increasing the efficiency of mathematics education (p. 4).

The difficulties intrinsic to mathematics, together with didactical errors in teaching and curricular flaws, might be considered by some to be the main reasons for students' failure in mathematics (Krygowska, 1975). The traditional Polish approach toward problems of mathematics education has focused on the subject matter content of mathematics, ways that teachers can express the scientific knowledge in a form that can be applied at school, the cognitive aspects of learning mathematics (with special emphasis on students' errors and teachers' reactions to them), and the metacognitive strategies students choose when facing the problem. However, contrary to commonly held views, mathematical activity is not purely cognitive (Schoenfeld, 1983; Op't Eynde, De Corte & Verschaffel, 1999; Liljedahl, 2005). Thus, it is worth investigating the multidimensionality of some other aspects of mathematics education as well. During the last decades, many researchers have paid increasing attention to *affect*. The basic structure of the affective domain is comprised of emotions, attitudes, beliefs (McLeod, 1992) and values (DeBellis & Goldin, 2006):

> Affect is symbiotically related to learning in mathematics education – students' beliefs, attitudes and emotions influence their learning in mathematics classrooms, and conversely, students develop mathematical beliefs, attitudes and emotions as they are engaging in the activities of the mathematics classroom. (Grootenboer & Marshman, 2015, p. 30)

Affect has been interpreted as

> a system of representation, encoding information about the external physical and social environment, mathematics, cognitive and affective configurations of the individual, cognitive and affective configurations of others (De Bellis & Goldin, 1999, p. 37).

One of the most important aspects of affect is meta-affect which refers to:

> affect about affect, affect about and within cognition about affect, and the individual's monitoring of affect through cognition (thinking about the direction of one's feelings) and/or further affect (DeBellis & Goldin, 2006, p. 136).

Just as metacognition stands in relation to cognition, so meta-affect stands in a direct relationship to affect. According to some researchers (McLeod, 1992), in order to increase the impact that research has on the effectiveness of mathematics education, major emphasis should be placed on affect.

2. The Role of Emotions and Beliefs in the Learning of Mathematics

Emotions and beliefs represent opposite ends of the affective continuum. Emotions are described in terms of rapidly changing states of feelings. They are typically locally and contextually embedded (DeBellis & Goldin, 2006). Being experienced either consciously or nonconsciously, emotions range from mild to intense. McLeod (1992) describes them as "hot" components of affect. We all experience a wide range of emotions, and as "the states of emotional feeling carry meanings for the individual" (DeBellis & Goldin, 2006, p.133), we may differ in the way we interpret our states. For example, two people experiencing a physiological arousal can assign different meaning to it. One might find arousal to be a signal that the problem he or she is dealing with is a challenging and non-trivial one, whereas somebody else might interpret the arousal as an indicator of threat. Jamieson, Mendes, Blackstock & Schmader (2010), show that our interpretation does influence the results of our actions. In their study, a group of students were told that the arousal they feel before taking an exam is a sign that the body is mobilizing resources to meet the task demands, and that the arousal actually helps to focus and do well on the exam. On the test these students achieved higher scores than their peers from the control group, who were not given this information, although the abilities of the two groups were comparable. This finding supports what DeBellis & Goldin (2006) had already suggested, that developing powerful meta-affective structures can turn out to be a key to unlock mathematical power in learners.

In contrast to emotions, beliefs are often considered to be rather stable and "cold". There exists no specific definition of beliefs that the whole research community agrees on. This problem has been a subject of an ongoing discussion (Furinghetti & Pehkonen, 2002; Grootenboer & Marshman, 2015). The stability of beliefs has also been called into question (Liljedahl, Oesterle, & Bernéche, 2009). In this paper, I focus on the learners' mathematics related beliefs. These are understood to be what the individual holds to be true about the self (in relation to mathematical activity), the nature of mathematics as a scientific discipline, and the learning and teaching of mathematics as a school subject (Underhill, 1988; Pehkonen, 1999; Lester, 2002; Goldin, 2002). This paper is also built on the assumption that belief systems can undergo an effective process of change.

People in general show a tendency to formulate beliefs on the basis of their experience. The same phenomenon occurs in school settings. For example, a student experiencing some recurring negative emotions while doing mathematics may come to one of the following beliefs: I am a humanist, I do not have a "mathematical brain", mathematics is irrelevant, there's no point in learning this subject, and so forth (Pieronkiewicz, 2015). These beliefs might be considered as defense mechanisms against mathematics (Nimier, 1993), and their role then would be to ease the tension and remove the source of frustration. By providing a reasonable explanation of facts or events that would otherwise be unbearable, beliefs help to sustain the state of equilibrium. One role of stable beliefs, thus, is to reinforce defenses against pain and hurt (Goldin, 2002).

A number of examples of students' mathematics related beliefs can be found in (e.g., in Lampert, 1990; Schoenfeld, 1992; Kloosterman, Stage, 1992). The exemplary list of "Typical student beliefs about the nature of mathematics" cited below comes from Schoenfeld (1992, p. 69)

> Mathematics problems have one and only one right answer.
> There is only one correct way to solve any mathematics problem – usually the rule the teacher has most recently demonstrated to the class.
> Ordinary students cannot expect to understand mathematics; they expect simply to memorize it, and apply what they have learned mechanically and without understanding.
> Mathematics is a solitary activity, done by individuals in isolation.
> Students who have understood the mathematics they have studied will be able to solve any assigned problem in five minutes or less.
> The mathematics learned in school has little or nothing to do with the real world. Formal proof is irrelevant to processes of discovery or invention.

As noted by Kloosterman and Stage (1992), students who believe solving a problem should take no longer than five minutes are more willing to give up when it takes them more time. Students who believe there is always a step-by-step procedure leading to the solution think their role is to figure that rule out and simply apply it. Many adolescents consider themselves not good enough in mathematics to be creative in the field, and for this reason they just choose to follow prescribed procedures without even trying to understand the justification behind them. These students find learning mathematics by rote, memorizing and following rules given by the teacher safer and more adequate to their self-perceived abilities. Kloosterman and Stage (1992) suggest that good counterexamples might contribute to changing this misleading and maladaptive beliefs. For example, a student who thinks he or she is not capable of solving a task when it requires much time, should experience success in solving a time-consuming problem to see that his previous conviction was wrong.

Students' experiences in the classroom shape their mathematics related beliefs, but at the same time, the beliefs pupils hold have a powerful – and often negative – impact on their behavior and the way they learn and attempt to use mathematics (Schoenfeld, 1992). As noted by Pehkonen (1994, p. 3), those

> who have rigid and negative beliefs regarding mathematics and its learning easily become passive learners who emphasize remembering more than understanding in learning.

Beliefs have been called *a hidden variable* in mathematics education (Leder, Pehkonen & Törner, 2006). As long as people are not aware of the beliefs they hold, it is unlikely they will change the maladaptive ones. Thus, "if we want to reflect on our behavior, and perhaps to change it" (Vinner, 1999, p. 146), it is necessary to bring the hidden variable into the light of individual and societal awareness. That is why it is important to talk about beliefs explicitly, also in the classroom. Beliefs built on negative emotional experiences in doing mathematics contribute to the rising and sustaining of negative attitudes toward this subject (Pieronkiewicz, 2015). By knowing a student's story with mathematics and delving into his or her mathematics related belief system, we may get some better understanding of the phenomenon known as math anxiety.

3. Math Anxiety from the Affective Perspective

Many researchers have investigated the phenomenon of math anxiety (MA). Due to space limitations, I shall not provide a description of MA here, but instead, I refer the interested reader to the existing literature (e.g. Lazarus, 1974; Tobias, Weissbrod, 1980; Ashcraft, 2002; Varsho, Harrison, 2009; Park, Ramirez, Beilock, 2014). Below I present a few findings from selected research reports that potentially might contribute to future educational interventions aimed toward helping the low-achievers to become successful math learners. A more extended discussion of the problem is presented elsewhere (Pieronkiewicz, in press).

The first finding here provides evidence that math anxiety occurs in the form of neurological responsiveness of the human body. Using brain imaging procedures Young, Wu & Menon (2012) found that the brains of students with identified MA show hyperactivity in the right amygdala region, which plays a key role in the non-conscious processing of emotion, as well as in the hippocampus, which is crucial for storing our memories and connecting them to our emotions. Researchers also showed that MA is associated with reduced activity in brain regions supporting working memory and numerical processing (the posterior parietal and dorsolateral prefrontal cortex). Another finding (Lyons, Beilock, 2012a, 2012b) is that students' brain activity is induced by the anticipation of having to do math, not by actually doing math. Moreover, anticipation of doing math activates those regions in the brain which are responsible for visceral threat detection, including physical pain. However when one begins doing what one is afraid of (in this case doing mathematics), the neural activity of the brain changes in such a way that both unease and anxiety are reduced. The researchers conclude that "it is not that math itself hurts; rather, the anticipation of math is painful" (Lyons, Beilock, 2012b). This finding suggests that the experience of pain depends on the psychological interpretation we attribute to our *anticipated* mathematical activity, rather than the task we are actually dealing with. For future educational interventions to be effective, Lyons and Beilock (2012a), suggest "emphasizing control of negative emotional responses to math stimuli (rather than merely additional math training" (p. 2102). We find similar ideas expressed by Moscucci (2007). Also DeBellis and Goldin (2006) emphasize that the most important goal in mathematics education is *not* to "eliminate frustration, remove fear and anxiety, or make mathematical activity consistently easy and fun" (p. 137), but rather to teach the students how to transform each emotional challenge, into productive experiences supporting learning and further development.

The aforementioned findings lead to two hypotheses I would like to formulate explicitly in this paper. The first of them is the following:

HYPOTHESIS 1 *At least in the cases of some students, the root of their explicitly declared reluctance and performed math avoidance is fear.*

Negative attitudes toward mathematics serve then as defence mechanisms, helping to sustain the state of equilibrium. On the basis of research results reported elsewhere (Pieronkiewicz, in press) I recognize different levels of emotional depth in the structure of fear:

> fear of mathematics → fear of doing mathematics → fear of failure → fear of experiencing emotional pain → fear of letting oneself feel one's true feelings → fear of losing self-consistency (risk of disintegration)

On the surface and conscious level, students report experiencing fear of mathematics and mathematical activity (or reluctance toward mathematics and mathematical activity respectively). A deeper analysis of students' utterances reveals hidden layers of previously experienced pain and hurt, of which students still keep the affective memory. Negative experiences from the past strongly influence future expectations. Some students already predict failure, humiliation, and disgrace while anticipating mathematical activity (in other words, before they start dealing with any problem). They are able to report on these feelings and predictions. There might be, however, something more, a hidden agenda standing behind these reported kinds of fear, that students with insufficient insight cannot bring into awareness. Some psychologists state, that during their lifespan people learn to "avoid their primary emotions and often need permission to feel" (Greenberg & Rhodes, 1991, p. 47). Being afraid of losing self-consistency, people are highly motivated to avoid negative affect. The natural aim of the self is to remain stable, but paradoxically, it is through the process of destabilization that change and newness occur. The process of *disintegration* may be positive (Dąbrowski, 1979), but only if it brings a person to self-reflection, results in raising the need for some constructive changes, and eventually ends with reintegration on a more advanced level of self-awareness. In order to heal emotional pain, one needs to acknowledge it first and invite it up into awareness. From what Curtis (1991) reports, it seems that avoiding mathematics and using the "I don't like math" defence may not only keep the affective memory of yesterday's hurts alive, but also reinforce its powerful impact on today's challenges.

The second hypothesis I would like to formulate, says that:

HYPOTHESIS 2 *It is possible to change a person's affect in such a way that neither previous negative emotional experiences, nor maladaptive beliefs the person already holds, preclude the development of his or her potential.*

To be more precise in describing the change that enables a person to overcome affective limitations, I introduce the concept of *transgression,* on the basis of which the definition of this particular change is formulated.

4. The transgressive concept of man

The term *transgression* has different meanings depending on the context in which the word is used. In geology *transgression* denotes the spreading of the sea over the land. In genetics the term means the increase in growth, size, fecundity, function or other characteristics in hybrids over those of the parents. In both examples "to transgress" means "to cross some existing boundary or limitations". The concept of transgression has been brought into the psychology domain for the first time by Kozielecki (1987, 1997). To this psychologist a man is a self-directed, expansive creature who is capable of intentionally crossing the *boundaries* of what he is and what he owns, to become who he might be, and to obtain what he might possess. Kozielecki emphasizes the importance and the developmental character of crossing over personal boundaries and subverting limitations in everyone's life. Acts of transgression can be taken in one of four worlds of transgression: a) material objects – territorial expansion in the physical world, b) other people – expanding the control over other people but also altruism and extension of individual freedom, c) symbols – intellectual expansion; going beyond the information given, development of knowledge about the world and d) oneself – self-creation, self-development, unlocking one's potential. In that sense, transgressions may be of different kinds: psychological or historical, individual or collective, constructive or destructive, but also, on other level, it can be creative or inventive and expansive (e.g. material, interpersonal, intellectual expansions). Kozielecki observed that people take one of two kinds of actions: *protective* – oriented toward the maintenance of the *status quo* and *transgressive* – exceeding the boundaries and enabling the development of personality. The main characteristics of these two types of human activity are juxtaposed in the table below.

Protective actions	**Transgressive actions**
necessary: "I know I have to"	possible: "I know I am able to"
play key role in adaptation and survival	satisfy higher needs of a human being
regulated by the needs of deficit	regulated by the needs of growth
maintain the status-quo	bring forth a meaningful change
depend mostly on the external environment	depend mostly on personality traits
repeatable	non-recurring
planned and often predictable	spontaneous and harder to predict
accompanied by negative emotions, especially fear	accompanied by positive emotions, especially hope

Table 1. Main differences between protective and transgressive actions.

The concept of transgression is described more detail in (Pieronkiewicz, 2015).

5. Affective Transgression in the Learning of Mathematics

The concept of transgression can be applied both to the cognitive (Semadeni, 2015) and to the affective (Pieronkiewicz, 2015) domain. People are potentially able to reflect on their cognition, emotions, attitudes, beliefs and values. Thanks to self-reflection we are able to recognize some counterproductive or maladaptive elements of our cognitive and affective structure. When we find an element that is not working to our advantage, we may repair it, remove it and when necessary replace it with another one. Many people are afraid of such changes, as if the stable sense of themselves depended on the stable content. However:

> A person is not dependent on a stable form for a sense of identity but rather on a sense of self as the agent of experience with continuity over time. (...) Pathology emerges when the self-organizing process becomes stuck in organizing the experience of self in a rigid fashion. (...) This complex rigidification often occurs because of certain overlearned responses to recurring threatening environmental conditions in which the person, as a function of anxiety, repeatedly organizes himself or herself in a particular manner. In threatening circumstances in which the self is being damaged, the trauma of the situation appears to produce such anxiety that the person hangs onto the familiar sense of self as a source of security and self-protection and forms a sense of coherence around a sense of being damaged. In so doing the person loses the capacity for flexible and spontaneous organization. It is as though the risk of being the process that one is and entering varied forms of organization is too high because the consequences are unpredictable. A sense of control of one's world can only be maintained by being a particular familiar way. The advantages of being this self are that the experiences, even if negative, are at least known. (Greenberg & Rhodes, 1991, p. 43).

In our daily life we often see people who are afraid of going through a change. Change involves not only reorganization of the external structure we live in, but also sometimes a revolutionary modification of one's internal cognitive-affective network. We show a tendency to clinging onto the schemas we know very well, even if they are negative. We choose, what we are more familiar with. When we look from this perspective at students dealing with mathematics, we may better understand the phenomenon of math anxiety and its resistance to treatment. Often the loss of self-confidence in mathematical activity begins with occasional failure to understand mathematics. This may lead to a series of failures on mathematical tasks or tests. Sometimes these failures take place in front of the class, which itself is a very stressful experience. Many students confess they have been stigmatized by their teachers and assigned to the group of so called "weak" students. Taking the challenge of changing his status as a low-achiever, a student takes on the risk of another failure (hence pain), losing the sense of self-consistency (lack of internal and external stability), facing the unknown and the unpredictable. Taking the first step toward such a change requires a lot of courage, determination and even more. The necessary condition for the change to happen is that the student has insight into the emotions s/he has experienced so far and into the beliefs s/he has formulated since these experiences have begun. Such self-awareness needs to be accompanied by the will to change, and a deep conviction that the change is good and possible.

The intentional process of overcoming personal affective barriers that preclude one's mathematical growth and development is called affective transgression. The process is a psychological, individual and constructive transgression toward oneself. The schema below presents where in the structure of affect, transgression should intervene in order to reverse persisting negative patterns.

Learning mathematics, seen through the lens of the transgressive concept of a man, may become an activity leading not only to improvement of one's skills and increase in one's knowledge, but also to inner growth and personal development.

The teacher's role in supporting the process of affective transgression should not be underestimated. It is the teacher who can give his low-achieving students the

> opportunity to experience themselves in a new way, to discover that they can survive what is dreaded or feared, that they can once again experience themselves as agents of their experience, rather than as victims, and that they can trust in their own continuity regardless of the content of their experience (Greenberg & Rhodes, 1991, p. 44).

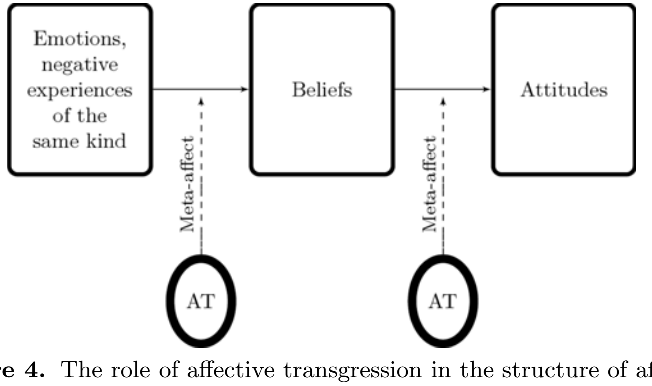

Figure 4. The role of affective transgression in the structure of affect.

6. Final Remarks

Instead of summarizing what has been presented in this paper, I would like to end up with a question: Can the future interventions that educators may take in order to help low-achieving students transgress their fears be successful?

I presume, there are two possible answers to this question. One, raising some doubts seen clearly from the teachers' perspective, expressed in the very sceptical spirit of Vinner:

> There are two essential conflicting elements in the human psychology which are active in the domain of teaching and learning mathematics: the need for meaning and the ritual schema; (. . .) there is no chance that one tendency will take over the other. The educators will continue their call for meaningful learning, whereas the masses of students will prefer the ritual (procedural) approach. (Vinner, 2000, p. 121).

The other one, representing the more optimistic and enthusiastic approach, is expressed in the words of Yeager and Walton (2011) who believe in the powerful potential of therapeutic interventions:

> Seemingly "small" social-psychological interventions in education – that is, brief exercises that target students' thoughts, feelings, and beliefs in and about school – can lead to large gains in student achievement and sharply reduce achievement gaps even months and years later. (p. 267).

Consideration of affective transgression as described here suggests the fundamental importance of the affective dimension in learning mathematics. To successfully apply this theoretical construct in school settings, we need a multidimensional affective profile of low-achieving students. In addition, we need to investigate moments when the change occurs, as well as moments when students refuse to go through with the process of transgression.

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About the relation between relationships and teaching and learning mathematics

Abstract. Mathematics education is a very vast field of research. But despite the quality and quantity of valuable research, the difficulties that students encounter in learning mathematics at school are widespread. This not only causes discomfort in children and young people, but also frustration in mathematics teachers. Moreover, one has to consider the social damage that results from it. Due to difficulties in mathematics in secondary school, many young people are deprived, firstly, of the opportunity to acquire skills in mathematics, and secondly, of the choice of pursuing degree programs that provide math courses. The author has accepted, since the early 2000s, the invitation of eminent mathematics education scholars to address parts of the research efforts to overcome the disconnection between the scientific research in mathematics education and the mathematics teaching practice. The aim of this presentation is to submit an approach to the problem of the difficulties in mathematics consistent with this objective. In particular, the author will deal with the issue of relationships, both in mathematics education research, presented in various meanings, and in mathematics classroom practice. Moreover, an educational path called MBSA will be presented, which is designed for the restructuring of the relationship of a person with mathematics, and which is widely used during both mathematics courses, in every level of education, and mathematics education courses, addressed to either future teachers of mathematics or ones already in service.

Key words and phrases: relationships, mathematics education research, affect, difficulties in mathematics.

AMS (2000) Subject Classification: 97C99.

1. Introduction

The II Interdisciplinary Scientific Conference "Mathematical Transgressions" offers researchers in mathematics education a wonderful opportunity to reflect on their research: its nature, its consistency, and the fallout on the quality of teaching and learning in math classes. Metasearch reflections are necessary, from time to time, for the researchers to remain concrete. Indeed, research, for many of the researchers, is a passion, and like all passions, often may get out of hand and become self-referential. It so happens that researchers fall in love with their research, thus it becomes the main purpose and makes them forget the fundamental purpose of their studies. In some areas of research, the matter is different: for example, research in mathematics can be very productive, even if it is not specifically aimed at a definite purpose. If George Boole had not enjoyed studying his algebras, maybe today the world would be different... but certainly it was not Boole's aim to contribute to computer science, nor to change the world! For mathematics education, it is another matter. It was created to study the learning and teaching of mathematics, which were perceived as issues worthy of further investigation. Let us remind that Alan Schoenfeld (2000) in "Purposes and Methods of Research in Mathematics Education" pointed out, as early as the year 2000, that mathematics education research has two main purposes: one pure and one applied. The pure one (basic science) aims at understanding the nature of mathematical thinking, teaching, and learning. The applied one (engineering) is focused on using such understanding to improve mathematics instruction. These purposes are deeply interdependent and they are equally important. Currently, in mathematics education research, it is rare to find papers purely focused on classroom practice for the application of theories, because often the line between theory and practical application is nuanced. In fact, it happens that ideas emerge from classroom practice and then they are specified and developed theoretically. Conversely, it happens that analysis and insight in theory lead to ideas for achievements in teaching. However, it is a fact that research in mathematics education is now very wide and prosperous. The II Interdisciplinary Scientific Conference takes place approximately a month after CERME 9 in Prague where 699 mathematics education researchers, coming from all over the world, presented their research results and compared their ideas. But despite such a great movement of ideas, what happens in our schools, in math classes? Very often, many students suffer because of mathematics! And even if it did not happen often, and even if there were few cases, it would still be too often and too many! And when students do not suffer because of mathematics, very often, too often, mathematics slips away from them without affecting their mind, let alone their hearts! What I mean is that the study of mathematics at school would lead to strengthening and restructuring mathematical thinking, which is a natural gift of every human being, firstly as a primate, also as a human being! There is no need for a domestic or international investigation to know that school mathematics very often causes suffering for the students as it is a well-known fact, and promoting research in this direction would be a waste of time and money. However, I can recall that in Italy, for example, it was stated (Moscucci et al., 2005) that there is a close connection between mathematics difficulties at school and school drop-out, in that the difficulties in mathematics are an 'endogenous variable' of school drop-out, a factor that is present inside the 'same educational institution' that fosters school drop-out. So, without reaching the limit of school drop-out, the difficulties that students encounter in school are a well-known problem while earning the discipline. This not only causes discomfort in children and young people, but also causes frustration in mathematics teachers. Moreover, the resulting social damage has to be considered with great concern: many young people, because of the difficulties in mathematics in secondary school, are precluded the choice of degree programs that provide mathematics courses. In conclusion, mathematics education researchers must acknowledge that in the face of such a high quality and quantity of research, in schools, teaching and learning mathematics does not work effectively. I recognize the freedom of the researcher to freely choose the objects of their research, but, in my opinion, very often, too often, research in mathematics education deals with refined questions that are understandable and interesting not even to all mathematics education researchers, but only to the experts of particular mathematics education research sectors. This is why I often compare this type of mathematics education research to the inflating a soufflé in culinary art! I call it "the mathematics education research of inflating a so ufflé"! This does not appear to conflict with my statement of appreciation of the freedom of a researcher. Indeed, I think it is very appreciable research, but it would be better if they were debated in working groups of the specific sectors and only then published, without taking up space in magazines that are a means of spreading ideas not only among non-experts, but also, for example, among mathematics teachers. I think that mathematics education researchers should address the problem of the difficulties in mathematics with greater determination, because too many of them are concerned with how to inflate the soufflé better, and too many students are 'starving for mathematics'! I have been dealing with difficulties in mathematics for 37 years and, since the late $90s$, I have started to follow an approach not only regarding the problem of difficulties in mathematics, but also the 'significance' of studying and teaching mathematics, which I call 'holistic'. This approach is based on the concept of a relationship in more than one way: this concept guided me towards putting forward two proposals to the scientific community which I am proposing on this occasion as well. My interpretation of "mathematical transgressions" regards both the approach to research in mathematics education and the teaching of mathematics at all levels and that, as I was saying, takes the form of two separate proposals, which are absolutely interdependent: one theoretical, and one experimental. I will not be going into the description of the theoretical framework within which the two proposals are situated, as that may be found in papers specifically focused on them (Moscucci, 2007; 2009; Moscucci and Bibbò, 2015). Here, I am mostly sharing an overall rationale behind the two proposals in order to fully appreciate the meaning and the potential of them and to have the opportunity of a fruitful discussion.

2. The difficulties in mathematics

2.1. Questions about difficulties in mathematics

Addressing the problem of difficulties in mathematics means to first ask the following questions: 1. What is the aim of teaching school mathematics? 2. Is school mathematics teaching functional with the aim stated or presumed? 3. What does it mean' to have difficulties in mathematics'? 4. Who has difficulties in mathematics and why? 5. How to overcome difficulties in mathematics? I do not want to answer to all these questions extensively right now, opting to give some indications and avoid any possible, though maybe useful, discussion about the meaning of school mathematics, because I believe that we all have a naive idea of school mathematics, which is independent from our country of origin. In my opinion, **the purpose of mathematics teaching is to promote the potential of the person**. Because this is not the specific purpose of mathematics, but of all school subjects, I point out that, in regard to mathematics, this aim might conduce every mathematics teacher to help their students be able to: **face problematic situations...** problem talking, problem finding, problem posing, problem solving, and **acquiring competences in mathematical languages** or, as Bruno D'Amore might say (for example, D'Amore et al., 2003), acquiring mathematical competences and acquiring competences in mathematics. This does not appear obvious in all countries and in all schools: In Italy, for example, the aim of mathematics teaching is especially focused on second competences and very little on the first ones. Acquiring the two types of competencies does not require different and separate teaching/learning paths: indeed, a teaching methodology inspired by various types of constructivism, gradually and properly selected by the teacher, is an exceptional opportunity for the realization of naturally arising synergies. This method of approach to the teaching of mathematics allows the teachers to organize work in math classes in order to really make the discipline a means of promotion of personal potential. In conclusion, I think it is an excellent way to reach the objective of making mathematics a means of personal promotion. Moreover, the choice of mathematics topics might be a good issue to discuss, but now and here this is not the right occasion.

2.2. The meaning of difficulties in mathematics: who has difficulties in mathematics?

In the light of what I have said so far about the aim of the teaching of mathematics, the term 'students, and people in general, with difficulties in mathematics' does not strictly mean' those who are not good at mathematics', but' those who are not able to use mathematics to promote their potential'. In my opinion, the reason why this happens so often at school is the traditional teaching method that is based, almost exclusively, on the transmission of knowledge without a high active participation of students in the classroom work. I have noticed, during my work in Italian schools, that many students with good school results have not mastered the basic concepts of discipline and, for example, in regards to algebraic language, they only have competence from a syntactic point of view, not from a semantic one. In conclusion, overcoming difficulties in mathematics does not mean to help someone who is not good at mathematics, but almost all the students, to make the discipline a means of personal promotion, and, as a consequence, to help most mathematics teachers, at least in Italy, to work with this aim. Now, if we agree to this goal, the fundamental question is the following, summarized as "How?", that is, "How can we realize this, firstly, as mathematics education researchers and, secondly, as mathematics teachers?" In the following, I will describe two proposals: one for empirical and developmental research (Moscucci, 2007) and one already presented at CERME 9, held in Prague in February 2015 (Moscucci and Bibbò, 2015), for theoretical research.

2.3. The relationship between the person and mathematics

Since the 80s, I have been interested in the problem of difficulties in mathematics and I approached it using a particular methodology that provides first, both temporally and substantially, the analysis, the consideration, and the revision of mathematics related beliefs and emotions of students with difficulties. In the late $90s$, this approach led to the realization of an educational path that has been used, since then, by me and by all the math teachers who took one mathematics education course of mine, both as prospective teachers of mathematics and as in-service teachers. This path deals, firstly, with the acquisition of the awareness of one's beliefs involved in viewing mathematics and self-confidence towards learning mathematics. Then, with the reworking of these beliefs to overcome any misconceptions, and the reworking of emotions linked in any way to mathematics. It is a serious rebuilding of the relationship of the person with mathematics. Indeed, during those years of working with the difficulties in mathematics, it emerged that students manifest the tendency to personify mathematics. This finding is supported not only by drawings related to mathematics (examples of representation: an old witch, an elderly lady dressed in black, grim, even a woman who hanged herself (Moscucci, 2008 a/b)), but also by interviews, questionnaires, and papers about mathematics, where students often refer to mathematics as one refers to a person (for example, "She's been haunting me for so many years…", "She has always been an obsession!", "Math is ugly and bad!", "It's a witch!", and note the female pronoun!). Moreover, the tendency to personifying mathematics is not exclusive to students with difficulties, as it also relates to students with no problems with mathematics, as well as students of specialization schools for higher education, or future teachers who often speak of mathematics as a person: "I have always agreed with mathematics", "She, for me, has always had an irresistible charm", "Mathematics has never betrayed me!". Therefore, the path that I am speaking of was called MBSA (Meta Beliefs Systems Activity) when presented at CERME 5 (Moscucci, 2007). It is "a path aimed to restructure the relationship of the person with math" and it has become a crucial point of all my courses, both of mathematics education and mathematics, and, as I said, it is routinely used by all of the teachers who attended a course of mine during their school work. I consider that, to teach mathematics profitably, we need, before initiating any disciplinary activities, to help students overcome those beliefs and any 'small trauma' involving mathematics. Regarding the students who do not have any 'small trauma', it is very difficult for them not to have any 'twisted beliefs', or absolutely unfounded beliefs about mathematics! At least in Italy it is so! For example, many people believe that either you are predisposed to learning mathematics or you are not, like a genetic matter, or at least a matter of familiarity! To remove this type of belief is not easy at all! The distinguishing characteristic of MBSA is its emphasis on awareness: the students are helped to autonomously become aware of their own belief systems and emotions and of their reasons and the way through which such beliefs were constructed. At the same time, the students are led to acquiring new knowledge regarding the issues of those beliefs in order to undermine the consistency of the beliefs themselves. This path usually has great success not only among the students, but also among teachers of mathematics, and it has been observed that the disciplinary paths of mathematics that they follow are definitely more profitable.

2.4. Conclusive consideration of the empirical and developmental proposal

The path of the restructuring of the relationship with mathematics is, in my opinion, the way to begin any path of teaching and learning mathematics by sharing a view of mathematics that is not distorted by the many widespread beliefs that hinder one's work in mathematics. Also, aside from the obvious purpose to overcome beliefs that have no scientific basis, this path allows for the construction of an atmosphere of sharing between the teacher and the students and it helps the teacher set a real community of learning. Therefore, besides tearing down the invisible wall that often stands between a student and mathematics, the path also helps in reducing the wall between a student and a teacher of mathematics, and this concerns not only students with difficulties, but all students. This leads us straight to the consideration of the other proposal, the theoretical one, but that has absolutely practical and actually considerable implications.

3. From the relationship with mathematics to relationships

The realization of the restructuring of the relationship of the person with mathematics leads very naturally to wondering about the nature and the construction of beliefs, and then, as a consequence, to wondering about the mind... that is, the 'residence' of the beliefs, the correlation with emotions and their nature... so the path is really a Trojan horse that allows us to bring a person to reflect deeply on important issues which are usually ignored by people and, in particular, at school by the teachers, much less mathematics teachers. On the other hand, the results of neuroscience of the 90*s* – for instance Damasio (1994) and LeDoux (1998) – do not allow researchers in mathematics education and in education to deal exclusively with the cognitive aspects of teaching and learning because, in the brain, the cognitive and emotional systems are absolutely interdependent. Moreover, my area of mathematics education research, Affect, considers, since the late 80s, beliefs, emotions, and attitudes as historical constructs. But, also, the teachers of mathematics, and, similarly, of all other disciplines, cannot deal exclusively with the cognitive shape of teaching and learning mathematics! The disciplinary competencies are a necessary condition, but certainly not a sufficient condition! Both in research and in teaching, other competences are needed, though not extensively, and not only designed to perform the function of a researcher or of teacher. Specifically regarding the use of the path, I never had any objections from future teachers or teachers in service of mathematics, because my teaching method is inspired by various types of constructivism, properly integrated, and the path that I suggest them to use with their students is exactly what I use with them, it just changes the mode of relating, as they are adults and their students not always are. Therefore, it changes the 'declension' of the activities of the path, but not the substance, as the pivots of the path are the same. Also, after I perform the full path, I look back with them, doing 'meta-education', that is, discussing with them the rationale of the path in its entirety, and of all of the activities within the path. Since the early 2000s, the MBSA path appeared to be successful. This success had two consequences. One was to drive me to explore the issues that had emerged during the path. The second was studying everything about beliefs and emotions, not only within mathematics education, but also drawing from other fields of study and research. All of this happened in a period when I began studying issues of quantum physics and neuroscience, in which I was interested for many years, but had never connected in any way to my research interests. Especially the studies of neuroscience seemed to be related to the research I was doing. In fact, for example, I used the MBSA path and then I followed it with another path aimed at recovering the calculation skills of individuals with mental retardation, and I obtained very good results significantly faster than speech therapists using standard methods. I understood then that an approach based on a new perspective, on scientific contributions outside mathematics education, could really make a change in my work. So there might be substance in finding the rationale and understanding the reasons behind the successful path, and maybe implementing it. I defined this new approach to the problem of the recovery of mathematical competences and of competences in mathematics as 'holistic'. I was immediately fascinated by the possibility of combining my studies with mathematics education. The enthusiasm grew each time I discovered a link. The strongest link that I soon discovered was the centrality of the concept of a relationship. I had already observed that in regards to the implementation of the MBSA path, an important variable was interpersonal relationships: the egg of Columbus? Maybe, but I thought there was something that I perceived clearly which was not so clear from a scientific point of view and that certainly deserved to be deepened. Since the early 2000*s*, it was clear that the success of the path of restructuring the relationship with mathematics was strongly linked to the quality of the relationships built by the teacher who made use of the path. However, after about ten years, the problem of the analysis of this quality began to take on a clearer connotation: it is likely that my mathematical formation got along with the theories of psychology from which I had been borrowing until then. They all seemed absolutely self-referential. In the late 2000*s*, it was clear that the relationships were not as simple as they seemed. I had the awareness of the nature of the interpersonal relationship being at least twofold: on the one hand perceptible, on the other unconscious. This clarity came to me in cooperation with one of my students, who later became my collaborator, Cecilia Bibbò. This collaboration has found its first synthesis in the presentation at CERME 9 in Prague of the proposal to the Affect community regarding the concept of a relationship as a new construct for this research sector of mathematics education.

4. About the theoretical proposal: relationships

4.1. Relationships in the Affect domain

The fact that the concept of a relationship has to be considered a real new construct of Affect is widely argued in the paper presented at CERME 9 (Moscucci & Bibbò, 2015). Here, it may be useful to summarize some points without engaging in the description of the theoretical framework. The historical constructs on which Affect research is based are emotions, beliefs, and attitudes (Mc Leod 1992), to which additional values were added at a later time (DeBellis & Goldin, 1997) and other issues were studied and recognized over time as motivation and metacognition (Hannula et al., 2004). These, however, were not recognized to be as important as the historical ones. Also, in my opinion, beliefs and emotions are the central ones and attitudes may be considered a natural consequence of emotions and beliefs which are the structure on which we engage all other concepts. The mind is structured through experiences and emotions and beliefs are the objects that constitute the pivots. Relationships play an important role in defining the quality of beliefs and emotions. Cecilia Bibbò and I are currently investigating this issue and we will soon submit the first paper about the subject for publication. Moreover, we believe that this issue is of great importance, not only in mathematics education, but in education in general. The human being is a social animal, and as such, its nature depends, in every sense, on its interpersonal relationships, from the relationships with the environment in general, and with everything which is a part of the environment. So, the importance of relationships within the Affect research field has a fourfold root. One is the concept itself that affects the teaching and learning mathematics as much as the other constructs, if not more. The second one, as I said before, is in the effects that relationships have on emotions, beliefs, and attitudes. The third root is borrowed from quantum physics and says that the nature of every object cannot be studied by extrapolating the object from the environment. In fact, the relationships of an object with everything with which it interacts not only determine its nature, but are an integral part of it. The fourth concerns the meta-theoretical aspect of Affect. Borrowing from the language of algebraic structures, we can express the situation briefly as follows. Relationships make the set of Affect constructs not just a set of constructs, but a real structure, that is, the relationships link the other Affect constructs, playing a role similar to that of a mathematical operation among the elements of the underlying set to define an algebraic structure.

4.2. About defining and characterizing relationships

Cecilia Bibbò and I did not propose a definition of relationships, because we believe that, when dealing with an issue of mathematics education, our scientific community might understand, without any problem, our proposal to deal with *Relationships* as a sort of primitive concept inside a sort of axiomatic theory and moving toward a characterization of the concept. And we have already begun this characterization, as I am going to explain. Cecilia Bibbò and I do not deal with relationships from a usual psychological point of view, but from an absolutely innovative one. Usually, a relationship is a set of information of interpersonal communication: verbal, gestural, mimics, tactile, postural, kinesthetic, etc., i.e. a set of interchangeable information, or the sensory information exchanged between two people. Cecilia Bibbò and I defined "rapport" as a set of information perceptible by the senses. Relationships are responsible for building the patterns of the interpretation of reality that determine the quality of emotions, and so they guide the ability of regulating emotions. Relationships are responsible for the building of beliefs about ourselves, others, and the environment itself, as well as the development of beliefs in general. All of this deals with the rapport between two people, but between two people there is also much more... There is a *hidden communication* too! Up till now, we knew from scientific results from the field of neurobiology that some of the existing neural structures let people exchange information without awareness. These neural structures refer to mirror neurons, and, perhaps, many other things yet to be discovered. Particularly, mirror neurons specialize in carrying out and understanding not just the actions of others, but also their intentions, behavior, and emotions, through direct feeling (Rizzolatti and Craighero, 2004). The existence and the functioning of mirror neurons proves that there is a 'link' between people, and we, as mathematics education researchers, cannot ignore such important scientific results that open new frontiers not only for mathematics education, but for education as a whole as well. The results of neuroscience concerning mirror neurons give the relationships a completely different role from the one they always had in scientific literature. In fact, relationships are not only an element of social and environmental interest: *relationships* are *a structural element of a person*! Cecilia Bibbò and I began to characterize the relationships, highlighting the double nature of relationships: we state that **a relationship is the sum of rapport and hidden communication**. What is more, we are absolutely aware that the hidden communication between people is to be deeply investigated and that the results regarding mirror neurons are only the small first step which proves that there exists hidden communication, but we have to investigate much more about the nature of it. The theoretical research in neuroscience about hidden communication is in its early stages. But this should not be a disincentive to study the relationships with this aspect of communication in mind. In fact, studies of relationships in learning environments can be a useful incentive for neuroscientists to study the subject. The interaction between neuroscience and education researchers, and, particularly, between Affect researchers and neuroscientists appears to be highly appropriate as of today and it is reasonable to say that the collaboration between researchers coming from such different research fields may allow for considerable progress.

5. Conclusions

Highlighting the double nature of relationships, that is, distinguishing **rapport** from **relationship** is the first step towards a characterization of relationships, but it is not a weak and uncertain first step: on the contrary, it has lead us to see relationships from a completely new point of view! Regarding Affect research and mathematics education research, it leads straight to an absolutely innovative approach to the learning and teaching, particularly, learning and teaching mathematics. In conclusion, I have dealt with: **1.** relationships as a new construct in Affect research; **2.** relationships as a double-nature-endowed link between two people;

3. relationships as a link between a person and mathematics; **4.** relationships as the relationship of a student with mathematics to overcome difficulties in mathematics and to make mathematics a means of personal promotion. These are my declensions of the concept of *relationships* in mathematics education. The aim of my presentation is twofold: firstly, to share my experience with the rebuilding of the relationship between a person and mathematics by using MBSA. Secondly, to bring attention of the research community in mathematics education to the concept of a relationship, not only its overall meaning as an interpersonal relationship, but as a sum of rapport and hidden communication.

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Research on Mathematics Education

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Structure-genetic Didactical Analyses. Empirical Research "of the first kind"

Abstract. In mathematics education, theories of teaching and learning based on disciplines different from mathematics ("imported" theories) are widely dominating the field. This imbalance greatly reduces the impact of mathematics education both on teacher education and on the teaching practice. In order to return to a balanced situation it is necessary to pay more attention to theories which are based on mathematics. As an example of such a "homegrown" theory, the paper presents the structure-genetic didactical analysis, the research method of mathematics education conceived of as a "design science".

A comparison of the papers published in journals and proceedings in the 1970s and early 1980s (see, for example, the pace – setting paper Krygowska, 1972) with the papers in the new millennium shows that over the past two decades the coordinate system of mathematics education has shifted massively *away from*

- the subject matter mathematics,
- the teaching practice and
- the critical examination of educational foundations concerning the subject,

towards

- qualitative and quantitative empirical studies of learning and teaching processes,
- the development and application of tests and
- theories of learning mathematics based on ideas imported from other disciplines.

Key words and phrases: Design science, substantial learning environments, didactical analysis, empirical research.

AMS (2000) Subject Classification: Primary 97C, Secondary 97D.

This shift consciously or unconsciously involved a break from the tradition of mathematics education. Nevertheless, this tradition is still alive. In recent decades a branch of mathematics education has developed that explicitly builds on the tradition of "subject matter didactics" as it has been common in the past in many countries. This "mathematics education *emerging from the subject*", as it has been called, continues to carry the teaching of mathematics and teacher education and has created a scientific basis of its own. The internationally known project "mathe 2000" may serve as an example (Wittmann, 2012). "Mathematics education *emerging from the subject*" constitutes by no means "didactics from the armchair" for which its predecessor had been criticized. On the contrary, it is supported empirically in its own way. Its specific feature is that it rests on theories of teaching and learning that are *implicit* in the subject mathematics itself. This will be shown in this paper, which is structured as follows: in the first three sections three central themes of the curriculum will be considered both from the position of the present mainstream in mathematics education and from the position of "mathematics education *emerging from the subject*". In the fourth section the research method of the latter, the *structure-genetic didactical analysis*, will be characterized and it will be indicated what can be achieved from this method.

1. Introduction of the multiplication table in the grade 2

In the curricula of many countries multiplication is introduced as "repeated addition" and the multiplication table is accordingly learned row by row. The last decade has seen a vivid discussion in the Anglo-Saxon countries about what multiplication is about. The empirical analysis of (Park & Núñes, 2001) fits into this context. The authors compared two hypotheses of concept formation for multiplication: multiplication as "repeated addition" and multiplication as a "schema of correspondences". What the latter means, however, remains unclear in that paper. It is likely that the authors allude to the interpretation of multiplication as a linear function: for a fixed multiplier c we have a mapping that assigns the product $x \cdot c$ $(= c \cdot x)$ to any number x. As a result of their research the authors arrive at the conlusion that "repeated addition" should not be used for defining multiplication, but only for calculating the results.

From the perspective of mathematics education *emerging from the subject* multiplication in grade 2 can be approached in the following way: multiplication is defined as "abridged" addition, as it is common in mathematics. For calculating the results it is natural to refer to the laws of multiplication: among the multiples $1 \cdot m$, $2 \cdot m$, $3 \cdot m$, $4 \cdot m$, $5 \cdot m$, $6 \cdot m$,

$7 \cdot m$, $8 \cdot m$, $9 \cdot m$ and $10 \cdot m$ there are four multiples that are trivial or easy to calculate:

$$\mathbf{1} \cdot m,\ \mathbf{2} \cdot m \text{ (double of } 1 \cdot m\text{)},\ \mathbf{10} \cdot m \text{ and } \mathbf{5} \cdot m \text{ (half of } 10 \cdot m\text{)}.$$

Other multiples from $3 \cdot 7$ to $9 \cdot 7$ can be derived from the easy ones by means of the distributive law:

$$\begin{aligned}
&3 \cdot m = \mathbf{2} \cdot m + \mathbf{1} \cdot m,\\
&4 \cdot m = \mathbf{2} \cdot m + \mathbf{2} \cdot m \text{ (or } \mathbf{5} \cdot m - \mathbf{1} \cdot m\text{)},\\
&6 \cdot m = \mathbf{5} \cdot m + \mathbf{1} \cdot m,\\
&7 \cdot m = \mathbf{5} \cdot m + \mathbf{2} \cdot m,\\
&8 \cdot m = \mathbf{10} \cdot m - \mathbf{2} \cdot m,\\
&9 \cdot m = \mathbf{10} \cdot m - \mathbf{1} \cdot m.
\end{aligned}$$

This approach has been elaborated by Arnold Fricke in his "operative didactics" and is widespread in German primary schools (Fricke, 1968). In the early eighties Heinrich Winter went one step further: In line with his general postulate to look at arithmetic from the point of view of algebra he suggested to use rectangular patterns of dots for representing multiplication (Winter, 1984). This proposal is also found in Courant & Robbins (1996: 3), a classic among mathematical textbooks, and in Freudenthal (1983, pp. 109-110). In (Penrose, 1994, pp. 51-53) it is even stated that rectangular patterns of dots are the most efficient means to explain what multiplication is about.

The preference of eminent mathematicians for these patterns underlines the fact that this representation of multiplication is not just a visual aid which has been invented for the purpose of teaching, but that is fundamentally interwoven in the epistemological structure of mathematics. The great advantage of this representation is that the commutative law, the associative law and the distributive law can be derived in an operative way and used in teaching (see, for example, (Wittmann & Müller, 2007, pp. 54-56). This is not possible with other representations of multiplication.

Later in the curriculum patterns of dots pass into the representation of a product as the area of a rectangle and this representation reaches up to the integral. It is a fundamental idea of algebra and calculus.

Comparison: What multiplication is about and how it should be introduced in the classroom, cannot be decided by means of empirical methods imported from psychology, but should be based on a sound mathematical and epistemological analysis. This, however, is not to say that empirical investigations of learning processes are superfluous (see section 3).

2. Designing a substantial learning environment for practicing long addition

While the first example deals with the didactical foundation of some topic the second example leads to the very core of teaching. The natural way to help learners to master some piece of knowledge or some skill is to offer them substantial learning environments that stimulate mathematical activities. Here the practice of skills plays a crucial role. Heinrich Winter introduced the concept of "productive practice" which means a type of practice in which contents and general objectives of mathematics teaching (mathematizing, exploring, reasoning and communicating) are combined (Winter, 1984).

In order to design a substantial learning environment for practicing long addition in our project "mathe 2000" we had to browse elementary mathematics for patterns that involve long addition. We had to check whether children's knowledge in grade 3 is sufficient for understanding and solving the intended tasks, for exploring, discovering and describing patterns and for explaining them by using familiar means with some support of the teacher.

Our analyses led us to the following learning environment that is based on the famous rule "casting out nines" (Wittmann & Müller 2012:, pp. 85).

The guiding problem posed to students is as follows:

> Form two three-digit numbers with the six digit cards 2, 3, 4, 5, 6, and 7 and add these two numbers.
>
> a) Find different results.
> b) Try to reach results as near as possible to 600, 700, 800, 900, 1000, 1100, 1200 and 1300.
> c) Try to find results between 900 and 1000.

The subtasks b) and c) are intended as hints for discovering the underlying pattern.

Guy Brousseau's theory of didactical situations provides a natural framework for the teacher in putting a learning environment into practice (Brousseau, 1997).

Here this theory can be applied as follows: In the first situation the problem is introduced to students, best by means of examples. In the second situation students work on their own, individually or in groups. The teacher serves as an advisor.

In the third situation the results are collected and compared. The teacher is free to add some more examples, and to give hints that stimulate students to discover the underlying pattern. Subtask b) is particularly helpful as the optimal results 603, 702, 801, 900, 999, 1008, 1098, 1107, 1197, 1206, 1296, 1305 reveal a striking pattern: The total of the digits of these numbers is 9, 18 or 27.

The results in subtask c) support these findings. Possible results are 900, 909, 918, 927, 936, 963, 972, 981, 990, 999.

A check with other examples will confirm this pattern. Of course some students will offer calculations with results that seem to violate this pattern. However, checks will reveal mistakes in the calculations.

In this way the conjecture is formed that for this problem only results are possible for which the total of the digits is a multiple of 9.

Situation 4 in Brousseau's classification requires the explanation of this pattern. The place value table with which students in grade 3 are familiar, serves this purpose perfectly (Wittmann & Müller, 2013: 120-121): Some examples are represented by means of counters on the place value table. It is interesting to note that in this context the total of the digits of number has a very concrete meaning: It denotes the number of counters that are necessary for representing the number on the place value table.

Fig. 1 shows two examples:

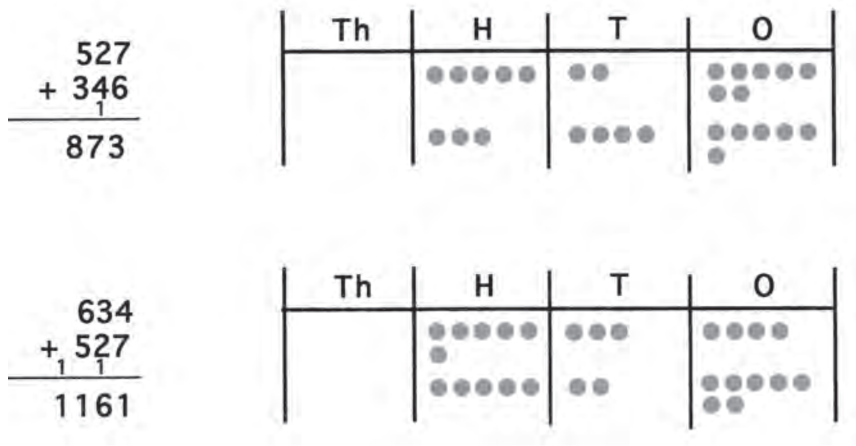


Figure 1.

In the first example $5 + 2 + 7 = 14$ counters are needed to represent the first number 527 on the place value table, and $3 + 4 + 6 = 13$ counters are needed to represent the second number 346. So $14 + 13 = 27$ counters are needed to represent the sum $527 + 346$. To execute this addition on the place value table means to push the counters in all columns together, and to replace 10 counters in the Ones column by 1 counter in the Tens column. Therefore 9 counters less than 27 are needed to represent the result 873, namely 18 counters.

In the second example again 27 counters are needed to represent the sum. We have a carry from the Ones to the Tens column and a second carry from the Hundreds to the Thousands column. According to the two carries the total of digits of the result 1161 is $27 - 2 \cdot 9 = 9$.

As in all examples 27 counters are needed to represent the sum the total of the digits of possible results must be 27, 18 or 9.

The final didactical situation is "institutionalization". Here the teacher's task is to summarize in a concise way what has been discovered. This might include the information that the operation of "casting out nines" is independent of the special numbers used here: For any sum of two or more numbers the sum of the totals of the digits of the numbers differs from the total of the digits of the result by a multiple of 9. The reason is that any carry involves a "loss" of 9 counters.

The teacher should also have in mind that this operative proof of the rule "casting out nines" is not an impasse, but that it can be continued later in the curriculum for deriving the divisibility rules (Winter, 1983).

Comparison: In this example the "home-grown" approach is unrivaled. It is obvious that theories of mathematics education imported from elsewhere, as well as empirical methods, are blunt when it comes to designing substantial learning environments. Only a thorough knowledge of mathematical structures and processes connected with curricular expertise will lead to solutions, and this knowledge is also essential for the teacher in doing her or his job.

3. Nets of a cube

Nets of the cube are a standard topic of mathematics teaching at the secondary level. In this section two approaches to this topic are compared.

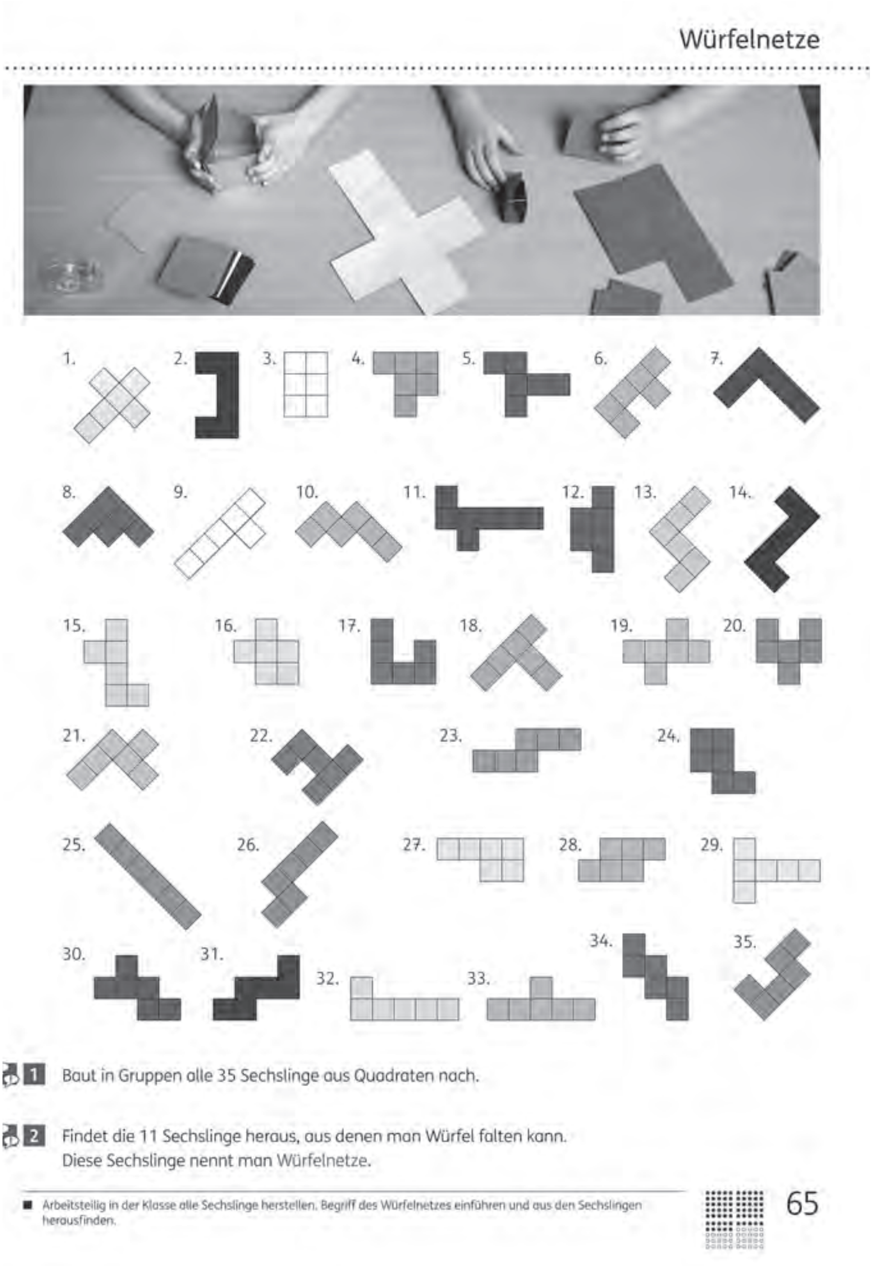
Susanne Prediger and Claudia Scherres have conducted guided clinical interviews with pairs of students in grade 5 (Prediger & Scherres, 2012). The objective of this study has been to investigate in some depth how students proceed when trying to find as many different nets as possible. The authors applied quite a number of empirical instruments in order to obtain a differentiated picture of the processes occurring during the collaboration. The results of this study are very complex and therefore cannot be summarized in short terms. For the following comparison two findings are relevant (Prediger & Scherres, 2012, p. 171):

1. Pairs of students can often exhaust their potential only through the intervention of the teacher.
2. The cooperation for exploiting the potential fully is enhanced when this cooperation is guided by mathematical considerations.

From the perspective of developmental research the first objective of a didactical analysis concerning the topic "nets of the cube" is to find out at which place of the curriculum students are in a position to respond to the requirements that certain treatment of this topic involves. At the very outset it should be kept in mind that any beautiful and important topic might allow for different approaches suitable for different places in the curriculum.

In the "mat 2000" curriculum nets of the cube are embedded in the fundamental idea of "dissecting and recombining figures", which is systematically developed along grade levels. An easy way of determining all possible nets is revealed in connection with polyominoes, a rich topic that was introduced by (Golomb, 1962) and elaborated for the primary level in (Besuden, 1984). A polyomino is a composition of congruent squares edge by edge. Polyominoes that are congruent are considered as equal. It is easy to see that there is only one domino (with two squares), but that there are two different triominoes (with three squares). Children in grade 3 easily find all 5 tetrominoes (with 4 squares) by adding one square to triominoes, and also all 12 pentominoes (with 5 squares) by entending tetrominoes. It is a stimulating task for kids to determine the 8 pentominoes similarly that can be folded into an open cube.

In a textbook for grade 3, the 11 nets of a cube are obtained in the following way (Wittmann & Müller, 2013, p. 65): The children are informed that it is possible to derive all 35 hexominoes by extending the 12 pentominoes. As this process would take too much time, the 35 hexominoes are provided by the teacher (Fig. 2) and the students are asked to find out which of these hexominoes are nets of a cube. In Fig. 2 the nets are arranged in five groups of 7 nets. This suggests forming five groups of students each of which has to make their 7 hexominoes with paper squares and sellotape and to investigate which ones can be folded into a cube. All five groups have to explain the reasons why some of their hexominoes do not produce nets. So in cooperation all 11 nets are determined through cooperation in a rigorous way.



Würfelnetze

1. 2. 3. 4. 5. 6. 7. 8. 9. 10. 11. 12. 13. 14. 15. 16. 17. 18. 19. 20. 21. 22. 23. 24. 25. 26. 27. 28. 29. 30. 31. 32. 33. 34. 35.

1 Baut in Gruppen alle 35 Sechslinge aus Quadraten nach.

2 Findet die 11 Sechslinge heraus, aus denen man Würfel falten kann.
Diese Sechslinge nennt man Würfelnetze.

■ Arbeitsteilig in der Klasse alle Sechslinge herstellen. Begriff des Würfelnetzes einführen und aus den Sechslingen herausfinden.

65

Figure 2.

An alternative approach at this level would be to start from the 8 pentominoes that can be folded into an open cube and to extend them to nets of a cube. However, as most nets can be derived from different nets of an open cube, it may be rather complicated to eliminate congruent nets.

In grade 5, the theme nets of a cube should be revisited. Again it seems appropriate to provide the students first with paper squares and sellotape and to stimulate them to find as many different nets as possible. Based on students' findings the teacher can guide the students to a systematic derivation of all possible nets. A natural way is to refer to the "addition principle" of combinatorics which consists of subdividing the set of combinatorial possibilities into subsets which are easier to manage. In the case of nets of the cube the maximum number of squares in a row is an appropriate criterion for a classification as is indicated briefly:

Case 1: *6 squares in a row*

No cube is possible as there are overlays and two faces remain open.

Case 2: *5 squares in a row*

Again no cube is possible as there is one overlay and one face remains open.

Case 3: *At most 4 squares in a row*

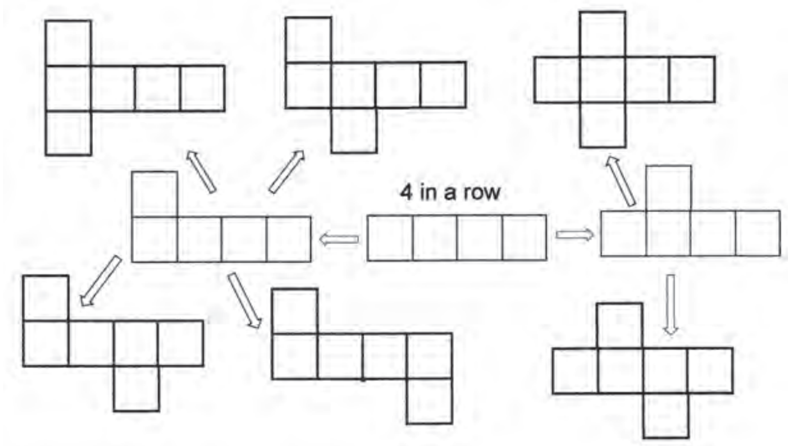


Figure 3.

First it must be found out where a fifth square can be added so that a net becomes possible. For each of the two possible positions of the fifth square the possible positions of the sixth square have to be determined. Some care is needed to eliminate nets that are congruent to nets that have been found before. Fig. 3 shows how to proceed stepwise starting from four squares in a row. The six nets determined in this way are drawn in bold lines.

Case 4: *At most 3 squares in a row*

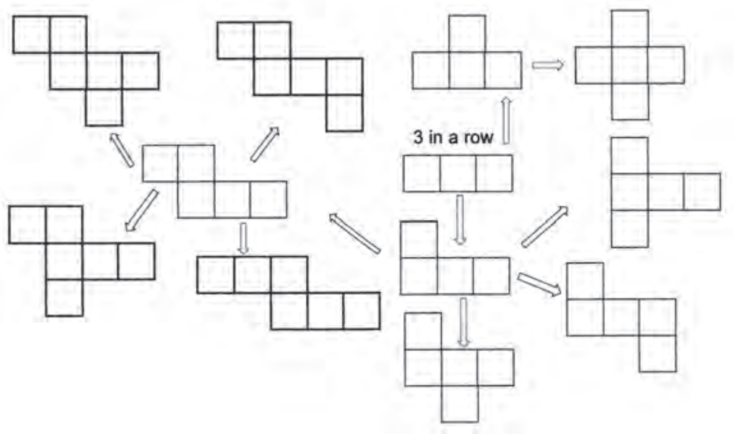


Figure 4.

In Fig. 4 no arrows are drawn away from the four pentominoes on the right. The reason is that the extensions of these pentominoes would result in nets that were already found.

Case 5: *At most 2 squares in a row*

In this case there is essentially only one way to get a net (Fig. 5).

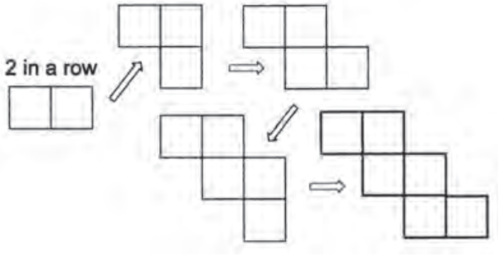


Figure 5.

It is obvious that this systematic derivation of all 11 nets of the cube is not easy. However, only means are used that are accessible to students in grade 5. With the assistance of the teacher, this learning environment is good to handle.

Of course it cannot be predicted how the investigation of this learning environment might develop in a certain class. Every interaction takes place under the particular circumstances of the class. However, a teacher who knows the mathematical background thoroughly is in a position to deal flexibly and productively with the contributions and ideas from the students. Based on their findings the teacher can introduce the classification. Different groups of students can investigate the three cases. In this way the complexity of the task is reduced to a reasonable level. The teacher can provide support where necessary.

Comparison: In this example the empirical investigation and the didactical analysis complement each other. Both are useful and instructive. There is no question that a teacher who has more insight into the processes linked to finding the various nets is more likely to interact with the students than a teacher who closely adheres to the mathematical structure and hardly leaves any room to the students. On the other hand, it is hard to imagine that a teacher who does not have a clear picture of the mathematical structure can organize a lesson solely with the spontaneous ideas of the students and with general pedagogical knowledge.

With respect to teaching and to teacher education, there are nevertheless significant differences between the two approaches. It is questionable if the "high resolution" instruments that have been employed in the empirical study by Prediger & Scherres (2012) can be communicated to teachers and students teachers in the time that is usually available in teacher education. It is also a question whether the results of this study can be integrated into teaching materials that work without the intervention of a teacher.

In contrast, the didactical analysis requires only a relatively small amount of time and can be well integrated into teacher education. The language that is used is simple and easy to understand. If the nets of a cube are included in both mathematical and didactical courses in an inquiry-based way there is a good chance that the metacognitive and cooperative skills that have been found as important in the empirical study can be acquired implicitly in these courses. This, however, is not to devalue empirical studies. The aim of this paper is to plead for didactical analyses as one tool of mathematics education without excluding other tools.

4. Structure-genetic didactical analyses

The approach of "mathematics education *emerging from the subject*" is based on the following assumptions:

1. Mathematical skills and techniques are acquired best in an active way under the guidance of mathematically experienced teachers. This refers to both teaching and teacher education. The practice of skills in its various forms plays a crucial role for successful learning.
2. The level of achievement that can be reached depends on the organization of teaching along fundamental mathematical ideas that are being revisited continuously. Only in this way is it possible to secure solid foundations for further learning and to brush up on prior knowledge. Also, only in this way it is also possible to provide mathematical structures as building blocks for modeling real situations. The development of curricula that are consistently and systematically designed accordingly and combine the orientation towards structures with the orientation towards applications is the central task of mathematics education.
3. Authentic mathematical activities in which heuristic plays a crucial role, are by their very nature social and communicative and *quite naturally* include theories of teaching and learning (implicit didactics). To make student teachers and teachers aware of these implicit theories by referring to their own mathematical experiences is the most direct and most efficient form of providing them with (explicit) didactical knowledge.

Against this background, didactical analyses as employed in the examples above are playing a fundamentally important role. This research method, which is the gold standard in mathematics education conceived of as a "design science", is an extension of the traditional "subject matter didactics". While the latter has been focused on the logical analysis of subject matter and too much linked to the "broadcast" method of transmitting knowledge from the teacher to the student, the extended method emphasizes both the genesis of knowledge over the grades and individual learning processes. In order to emphasize this wider perspective, the term *structure-genetic didactical analysis* is proposed for this extended method.

The above examples show that structure-genetic didactical analyses are linked to hard facts: to the mathematical practice in exploring, describing and explaining patterns on various levels, to the prerequisite knowledge of learners, to the objectives of teaching and to the curriculum. This is all *empirical material*. Therefore, the structure-genetic didactical analysis is an empirical method. Because of its nativeness it may well be considered as empirical research of "the first kind". The usual empirical studies are then empirical research of the "second kind". The assertion that only empirical studies of the second kind would provide "evidence-based models" for teaching and learning is untenable.

Structure-genetic didactic analyses are of primary importance in mathematics education for the following reasons:

1. They emerge from the *mathematical practice*, that is on *doing mathematics*, at various levels.
2. They foster an *active relationship* with mathematics as a living subject mathematics.
3. They are *constructive* and therefore absolutely essential for designing substantial learning environments and curricula.
4. They are natural guidelines for teachers, as they unfold the *implicit theories of teaching and learning* of mathematics, that is, as they "unfreeze" the "didactical moments frozen in the subject" (Heintel, 1978: 46).
5. They are *meaningful for teachers*, as the feedback from the field clearly demonstrates.

The examples in the first three sections show that structure-genetic didactical analyses take the following points into account:

– mathematical substance and richness in activities at different levels,
– evaluation of cognitive demands on students,
– curricular matching (with respect to contents and general objectives),
– coherence and consistency along the curriculum,
– curricular reach,
– potential for practicing skills (most important!),
– estimation of the expenditure of time.

Paradigms of structure-genetic didactical analyses are (Wheeler, 1963), (Freudenthal, 1983) and the developmental research initiated by Hans Freudenthal at the IOWO in the 1970s, the developmental research initiated by Nicolas Rouche at the CREM in Belgium, see for example (Rouche et al., 1996), as well as the work of Heinrich Winter, the German Freudenthal, in particular (Winter, 2015). These paradigms demonstrate that the development of mathematics education as a research discipline also depends on the design of conceptually founded substantial learning environments. Achievements in this direction have to be acknowledged as *results of research.*

In the context of this paper point 4 above is of particular importance and therefore deserves some elaboration. The idea that theories of teaching and learning are implicitly contained in the subject matter, and that therefore mathematics education is not completely dependent on imports of theories from other disciplines is by far not new. More than 100 years ago John Dewey has formulated this idea with a clarity that leaves nothing to be desired. In his paper there is a long enlightening section on the importance of the subject matter for teacher education (Dewey, 1977, pp. 263-264):

> Scholastic knowledge is sometimes regarded as if it were something quite irrelevant to method. When this attitude is even unconsciously assumed, method becomes an external attachment to knowledge of subject-matter. It has to be elaborated and acquired in relative independence from subject-matter, and then applied.
>
> Now the body of knowledge which constitutes the subject-matter of the student teacher must, by the very nature of the case, be organized subject-matter. It is not a separate miscellaneous heap of scraps. Even if (as in the case of history and literature), it be not technically termed "science," it is none the less material which has been subjected to method – has been selected and arranged with reference to controlling intellectual principles There is, therefore, method in subject-matter itself – method indeed of the highest order which the human mind has yet evolved, scientific method.
>
> It cannot be too strongly emphasized that this scientific method is the method of the mind itself. The classifications, interpretations, explanations, and generalizations which make subject-matter a branch of study do not lie externally in facts apart from mind. They reflect the attitudes and workings of mind in its endeavor to bring raw material of experience to a point where it at once satisfies and stimulates the needs of active thought. Search being, the case, there is something wrong with the "academic" side of professional training, if by means of it the student does not constantly get object-lessons of the finest type in the kind of mental activity which characterizes mental growth and, hence, the educative process. (...)
>
> Only a teacher thoroughly trained in the higher levels of intellectual method and who thus has constantly in his own mind a sense of what adequate and genuine intellectual activity means, will be likely, in deed, not in mere word, to respect to the mental integrity and force of children.

For teaching practice this view is of fundamental importance: The ancient Greeks understood '*theory* as *view*'. The Greek word for theory, ϑεωρία, is derived from ϑεωρεῖν, which means *viewing*, *regarding*, *observing*. In this original sense a *theory* provides a comprehensive view of some area that allows for acting purposefully in this area while taking the circumstances and contingences in this area into account. The natural theories of teaching and learning embedded in subject matter serve exactly this purpose: they represent practicable theories *for the teacher*, as they supply him or her with profound information or knowledge on which to base her or his actions. Whether it is to introduce children to multiplication, or to practice long addition, or to determine the nets of the cube; or to estimate students' prerequisite knowledge, to activate their thinking, to interact and communicate with them; or to interpret students' oral and written utterings, to assess their learning progress or to start remedial work – all this is essentially determined by the teacher's "comprehensive view" of the topic to be learned. That teaching does not proceed smoothly, that there are breaks and obstacles in the learning processes, that students make mistakes, have difficulties in understanding some points, forget what they have learned before, and so on. This knowledge is an essential part of the implicit theories of teaching and learning arising from *an active mastery of subject matter*.

What therefore counts most in teacher preparation is not an explicit didactical component (i.e., method courses), but the *mathematical* component, *given that* in this component mathematical activities are offered that stimulate and provide student teachers with relevant experiences in regard to learning processes, including learning difficulties, phases of confusion, confidence in overcoming difficulties and so on.

Mathematical courses organized in this way also provide the most effective *theoretical* basis for teaching. This is not to say that theories imported from other disciplines are of no use. They may be. This is also not to say that method courses are superfluous. Rather, both imported theories and method courses can significantly enhance structure-genetic didactical analyses. However, they should not replace them.

5. Conclusion

This paper is a plea for structure-genetic didactical analyses, the empirical research of the first kind. It must not be misunderstood as a plea against empirical studies of the second kind. On the contrary, such studies are indispensible, when new topics are to be introduced, for which no information on students' prerequisite knowledge is available, and when new approaches or new means of representations are used. Examples are the introduction of stochastics at the primary level or the use of digital media. Empirical research of the second kind is also very useful for investigating the processes more closely that occur when a learning environment is "staged" in the classroom. Of course these studies are all the more revealing and more meaningful, the closer they are attached to structure-genetic analyses.

It has also to be acknowledged that a wider perspective in mathematics education including imports from related disciplines significantly contributes to a better understanding of mathematics and therefore supports structure-genetic didactical analyses. In this sense the present author has greatly profited from Jean Piaget's genetic epistemology. It is no accident that the term "genetic" is a constituent of the term "structure-genetic didactical analyses".

In a position paper on the nature of mathematics education Heinz Griesel contended that his sense "didactical analyses" would not differ from the "logical analyses" of mathematics (Griesel, 1974). Heinz Steinbring rightly rejected this narrow view (Steinbring, 2011). With structure-genetic didactical analyses the situation is completely different. These analyses include logical analyses, it is true, however, they involve also knowledge about mathematical processes, about the curriculum, about students' prerequisite knowledge at different levels, and about the boundary conditions of teaching. A mere knowledge of (elementary) mathematics is by far not sufficient. To put oneself in the place of a child who takes his or her first steps in early mathematics, to look at the multiplication table with the eyes of a second grader, to find the nets of a cube with the means that are available to students at the secondary level, or to make the concept of a limit accessible to high school students, all this requires a special didactical approach and a special sensitivity for the genesis of knowledge and for the mathematical practice at the level in question.

Mathematics education has certainly been enriched enormously by contributions from other disciplines. Structure-genetic didactical analyses are nevertheless the key for developing mathematics teaching and teacher education. Without them mathematics education is in danger to degenerate into a self-referential system. Jeremy Kilpatrick's warning of the "reasonable ineffectiveness of research in mathematics education" should, thus, be taken seriously (Kilpatrick, 1981).

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The Goldilocks principle revisited: understanding and supporting teachers' proficiency with reasoning and proof

Abstract. Many teachers face difficulties with reasoning and proving, especially when they support their students'work with these mathematical practices. I outline the background to a planned development project in primary/lower secondary teacher education that seeks to alleviate these difficulties. I argue that the project needs to deal with reasoning and proving in problem contexts that are 'sufficiently close' both to the challenges teachers encounter in mathematics classrooms and to the practices of reasoning and proving in the discipline of mathematics. This is uncontentious, as much recent scholarship on mathematics teacher education argues for the need to balance school mathematics and academic mathematics. A more specific (and possibly more contentious) suggestion is that, in the case of mathematical reasoning, this means balancing "proving that" and "proving why" in ways that build on the mathematical complexities of tasks that are used in school mathematics. To make my argument I draw on a conceptual framework called Patterns of Participation (PoP). PoP views teachers' acts and meaning-making as their (re-)engagement in other past and present practices in view of the interactions that unfold in the classroom rather than as their enactment of reified knowledge and beliefs. I use PoP-interpretations of classroom episodes to exemplify both the challenges teachers face when dealing with mathematical reasoning and the tasks that may be used in mathematics teacher education. However, my paper is not an empirical piece in the usual sense, but an empirically informed theoretical essay that outlines the background to the development project.

Key words and phrases: Mathematical reasoning, mathematics teacher education, Patterns of Participation.

AMS (2000) Subject Classification: Primary 97B10, Secondary 97B50.

1. Introduction

The fairy tale of Goldilocks is about a girl, who wanders in the woods and comes to a cottage inhabited by three bears. She enters and finds three bowls of porridge, one of which is too hot, while another is too cold. The last one is just right, and Goldilocks eats it all.

The Goldilocks principle, i.e. the need to avoid both of two extremes and settle for some middle ground, has been used about different situations in education and educational research. A banal example is that tasks for students should neither be very easy nor excessively difficult, as both situations lead to lack of involvement and intended student learning. Katz and Raths (1986) provide a very different example and refer to the Goldilocks principle to suggest that 'conceptual size' may explain why some recommendations for teachers and teacher education are too small or specific to be used sensibly (e.g. drill of particular teaching skills), while others are too large to guide action. Drawing on Katz and Raths, Kagan (1990) suggests that 'teacher cognition' is too comprehensive and ambiguous construct to be valuable for understanding or supporting teacher proficiency, while lists of behavioural objectives are unhelpful for the opposite reason.

In this paper I use the Goldilocks principle for yet another purpose, namely to discuss how to understand and address the difficulties many teachers have with reasoning and proving R&P in their classrooms. Specifically, I present the background to a development project on (R&P) in elementary mathematics teacher education. The argument is that there is a need to balance *proving why* and *proving that*. However, I do not merely make the possibly uncontentious point that some such balance is needed. Rather, I use a conceptual framework called *Patterns of Participation* (PoP) in order to provide a (probably more contentious) argument for why this is so and for how the balance may be conceived.

I begin with a summary of the discussion of teachers' knowledge of mathematics, followed by one on the recent interest in reasoning and proving in mathematics education. I then present key elements of the PoP framework, before referring to a study of a novice teacher, Larry, which will function as the empirical reference point for my presentation. However, this is not meant as an empirical paper, but as an empirically informed theoretical essay in which I draw on the case of Larry at his school, Mellemvang, to make my point.

2. Mathematics for teaching

Shulman and his colleagues pointed to a blind spot in many teacher education programs of the 1980s, namely the lack of attention to the contents of instruction (Grossman, Wilson, & Shulman, 1989; Shulman, 1986, 1987). They put subject matter issues back on the agenda by including three categories of knowledge directly related to the contents of instruction in a seven-category scheme of teacher knowledge: content knowledge (CK), curriculum knowledge, and pedagogical content knowledge (PCK). Content knowledge includes both substantive and syntactic structures, the latter being "the ways in which truth or falsehood, validity or invalidity are established" (Shulman, 1986, p. 9). In mathematics this is closely related to reasoning and proving. While CK in Shulman's framework is similar to the mathematical knowledge others have, if they are educated in the subject, PCK is special to the teaching profession. For the most common topics and ideas of a subject it includes

> the most useful forms of representation of those ideas, the most powerful analogies, illustrations, examples, explanations, and demonstrations – in a word, the ways of representing and formulating the subject that make it comprehensible to others . . . [and] the conceptions and preconceptions that students of different ages and backgrounds bring with them to the learning of those most frequently taught topics and lessons.
>
> (Shulman, 1986, p. 9).

Since Shulman introduced his scheme, large numbers of studies in mathematics education and beyond have focussed on and further developed his three content related categories (e.g. Ball, Thames, & Phelps, 2008; Kunter et al., 2013; Ma, 1999; Rowland, Turner, Thwaites, & Huckstep, 2009). Such studies generally build on investigations of the mathematical challenges that prospective or practising teachers face in their classrooms or of how deep knowledge of the mathematics taught relates to the ways in which teachers solve tasks from school mathematics or support student learning. In different ways they convincingly argue that significant parts of teachers' content preparation need to be related to the tasks of teaching. It is, then, not only teachers' PCK that is special to the profession, but also parts of their CK. In what follows I refer to constructivist interpretations of CK and PCK as *Teachers' Mathematics* or TM-frameworks. Building their recommendations on such frameworks *the Mathematics Learning Study Committee* says that while there is little evidence of causal links between teachers' knowledge of academic mathematics and their students' mathematical learning, teacher education courses "that reflect serious examination of the nature of the mathematics that teachers use in the practice of teaching do have some promise of improving student performance" (National Research Council, 2001, p.375).

Studies of teachers' knowledge generally focus on key content areas such as number, algebra, or geometry. However, over the last few decades more emphasis has been attributed to the role in school mathematics of mathematical processes such as problem solving, communicating, modelling, and reasoning (Common Core State Standards Initiative, 2010; National Council of Teachers of Mathematics, 2000). In this paper I present a PoP perspective on how to support prospective teachers' proficiency with reasoning and proving and with teaching them. Before outlining the PoP framework, I present aspects of the literature on reasoning and proving in school mathematics.

3. Reasoning and proving in school mathematics

In what has become a classic in mathematics education research, Lampert (1990) argues that school mathematics is generally not in line with key characteristics of mathematics as a discipline. Mathematical discourse, she says, is "about figuring out what is true, once the members of the discourse community agree on their definitions and assumptions" (p. 42). In contrast, school mathematics is dominated by a communication structure in which the subject degenerates into a rule-following activity in which truth and falsehood is determined by the teacher. This suggests that school mathematics is generally characterised by what Harel (2007) calls authoritative proof schemes, if proof is an issue at all, while mathematics is dominated by deductive ones. Much work in mathematics education, including much of Lampert's own work (e.g. Lampert, 2001) may be read as attempts to remedy this situation and understand and develop participation structures that bring school mathematics more in line with the discipline, while acknowledging the need for the students to see the discourse as meaningful. In spite of these attempts, Lampert's comment about school mathematics appears worryingly valid a quarter of a century after it was made.

The understanding of mathematical discourse in Lampert's deceptively simple statement corresponds to the priorities in more recent discussions of school mathematics, especially as they relate to reasoning and proving (Common Core State Standards Initiative, 2010; National Council of Teachers of Mathematics, 2000). Elaborating on these latter processes, NCTM suggests that R&P is conceptualised as a cycle of exploration, conjecture, and justification (National Council of Teachers of Mathematics, 2008).

The functions of proving as well as the specific character of and relationships among the elements of the reasoning-and-proving cycle have been discussed extensively (e.g. A. J. Stylianides, 2007; G. J. Stylianides, 2009). Yackel and Hanna (2003) argue that proofs are used for different purposes and that the most powerful ones in education are explanation and communication. This is so, as "chains of logical argument do not function as proofs unless they serve explanatory and communicative functions" (p. 228). Similarly, Hanna (2000) argues that "in the classroom the key role of proof is the promotion of mathematical understanding" (p. 5). And Hersh (1993) says that in mathematics proofs are to convince, but that in mathematics education they are to explain. The joint emphasis in these and other studies is that if compared to mathematics there is – or should be – a shift of emphasis in mathematics education from *proving that* to *proving why*.

Part of the background to this shift is that in schools proving often degenerates into a ritualistic rule-following that bears little resemblance to reasoning and proving in mathematics (Rowland, 2002). This reflects at least in part that teachers, not least in primary and middle school, have significant problems themselves with reasoning and proving, let alone the difficulties they face with facilitating their students' engagement with these mathematical practices (Reid & Knipping, 2010). One suggestion for how to remedy this situation is to use generic examples (Rowland, 2002) or 'single-case key idea inductive arguments' (Morris, 2007) to *prove why* in mathematics teacher education. Rowland refers to a paradigmatic generic example, the procedure for finding the sum of the first 100 integers by reordering the addends as $(100+1)+(99+2)+\ldots+(50+51)$. The procedure, normally attributed to Gauss when he was still a child, is generic in the sense that it easily extends to any other positive integer, n, and 'proves' that $1+2+3+\ldots+n=(n+1)\cdot\frac{n}{2}$. The generic argument, then, requires one to think of the general while operating on the specific.

There is ample evidence that generic examples have educational potentials for *proving why*, which are often left unexploited in mathematics classrooms. I offer an example from a grade 5 classroom as an illustration. However, inspired by the same teaching-learning sequence, I also suggest that if teachers are to capitalise on the potentials for mathematical reasoning in investigative classrooms, they need to be able to engage in other forms of proving, also some that focus on *proving that*. The examples are from a study conducted some years ago with a novice teacher, Larry (Skott, 2009). I no longer have access to the video recordings of this teaching-learning sequence, and what follows is a reconstruction based on transcriptions and my field notes. However, the details of the particular interactions are somewhat inconsequential for my present purposes. What matters are the opportunities that arise for the students to engage in mathematical reasoning, how the students may be supported in making the most of these opportunities, and if and how the episodes may inspire work with reasoning and proving in mathematics teacher education.

4. Patterns of participation (pop)

PoP focuses on how individuals draw dynamically on their prior experiences and reengage in other past and present practices in view of the meaning they make of the ones that unfold at the instant (Skott, 2013, 2015b, in press). This sets PoP at odds with TM frameworks, which are generally informed by acquisitionism, that is, by a metaphor for learning and knowing that "make[s] us think of knowledge as a kind of material, of the human mind as a container, and of the learner as becoming an owner of the material stored in the container" (Sfard, 2008, p. 49). According to this view, to know is to possess reified mental constructs that are stable across time, contexts, and activity. Radical constructivism may be regarded as a paradigmatic example of acquisitionism, as it is based on the premise that "knowledge, no matter how it be defined, is in the heads of persons" (von Glasersfeld, 1995, p. 1).

Over the last few decades acquisitionism has been challenged by studies that adopt more, or at least differently, social and processual perspectives on human learning and knowing. One such approach is developed by Sfard (2008). Drawing on Vygotsky (1978, 1986), she uses the term of commognition to link communication and cognition and suggest that thinking is nothing but internalised communication. To learn mathematics is, according to Sfard, a matter of engaging more proficiently in a mathematical discourse characterised by its vocabulary, its visual mediators, its "endorsed narratives" (i.e. results ranging from the outcomes of arithmetic operations to theorems in advanced mathematics), and the ways in which the narratives are substantiated.

While Sfard focuses on well-structured cultural practices, most notably mathematics, studies in social practice theory investigates learning in relation to social processes that unfold for instance among members of Alcoholics Anonymous (Holland, Skinner, Lachicotte Jr, & Cain, 1998), girl scouts in the US (Rogoff, 1995), tailors in Liberia (Lave & Wenger, 1991), and claims processors at an insurance company (Wenger, 1998). Key questions in these studies are what the characteristics are of the practices associated with the different social configurations, how such practices evolve, and how people come to behave and see themselves (or not) as members of the communities in question. From this perspective learning is not a matter of gaining ownership mental constructs or procedures, but of participating differently in social practices. The contrast to the premise of radical constructivism (cf. the quotation from Glasersfeld above) is apparent as a community-of-practice perspective "starts with this assumption: engagement in social practice is the fundamental process by which we learn and so become who we are" (the introductory note to Wenger 1998). For teachers of mathematics the question becomes how a novices learns to participate in the social practices that emerge at the schools where they are employed.

Symbolic interactionism also adopts a participatory approach (Blumer, 1969; Mead, 1934). According to Blumer, the meanings of the objects that humans act towards emerge in interaction, as we take the attitude of others to ourselves. As we act, we see ourselves through the eyes of immediate interlocutors and other individual or generalised others, present or absent. For example, the teacher may adjust the acts of teaching as she (fore)sees the lifted eyebrows or other reactions of the students; mentally reengages in a discourse on student learning at her college; or relates to the priorities of her colleagues as espoused in a recent staff meeting. The meaning of the situation may change for the teacher as the classroom interaction unfolds, and her contributions to the interaction are not based on stable, mental entities (knowledge and beliefs), but on her shifting interpretations of the situation at hand.

PoP draws on these three participatory approaches, but also differs from each of them. It seeks to understand how teachers learn to participate in the practices that unfold in their classrooms, but rather than focusing exclusively on classroom processes, PoP-studies relate such participation to shifts and changes in other practices that the teachers' draw on in the process. In the case of Larry at Mellemvang (see below), the question is how Larry's contribution to emerging classroom practices relates to genuinely mathematical practices; to the reform discourse as promoted by his teacher education programme; to discussions with his fellow students when he was still at college; to his interactions with his colleagues at Mellemvang; and to his tales of the educational priorities at the school. Rather than investigating one particular practice, e.g. in a classroom, PoP focuses on Larry and how he simultaneously engages in a range of different practices that all play a role and mutually transform each other in classroom interaction. In this sense and phrased in more general terms, PoP re-centres the individual in participatory accounts of learning (Skott, 2014).

The question of the present paper may now be rephrased as how mathematical reasoning may become a significant, subject specific practice for novice teachers to draw on in instruction. It is not the premise of the paper that R&P should necessarily be omnipresent in mathematics classrooms, even though it may be regarded as the *sine qua non* of mathematics. There may be many reasons, content specific ones as well as others, why a teacher decides not to engage the students in R&P in a specific situation. It follows that the episodes from Larry's classroom are not discussed in order to evaluate the teaching-learning processes from an R&P perspective. Rather, they are meant as a backdrop to considering a possible way ahead in mathematics teacher education, if teachers are expected to make reasoned decisions on when and how to engage their students in R&P and to be able to do so in situations, when this is deemed feasible.

5. Larry at mellemvang

Larry was selected for the study because of his commitment to his new profession and to the reform discourse as evidenced in a survey and an interview at the time of his graduation. Especially, he emphasised the importance of student investigations and the use of manipulatives. Six months later I observed Larry's teaching and conducted further interviews with him for $2\frac{1}{2}$ weeks at Mellemvang, the school where he got his first teaching position. Mellemvang is a traditional private school that emphasises student performance on standardised tests, and Larry uses a fairly old textbook with a similar emphasis. In the first interview, i.e. within the first week after taking up teaching at Mellemvang, Larry is concerned with the contrast between his own educational priorities and those of the school.

The data on Larry at Mellemvang were analysed using methods inspired by grounded theory (GT) (Charmaz, 2006). However, the coding methods, the constant comparisons, and the memo writing of GT were used without subscribing to the objectivist underpinnings often associated with them. The analysis shows that Larry and his students have developed an atmosphere in which the students often make their own conjectures and suggestions for possible relationships and patterns concerning the mathematical contents. This is the case also in first few lessons observed at Mellemvang, when Larry teaches a chapter in the textbook on perfect squares and cubes. Larry supplements the tasks in the book with the students' use of *centicubes*, $1 \times 1 \times 1$ cm cubes that are often used to teach place value. The cubes may be assembled to make geometric representations of squares and cubes.

A whole class session

In the second lesson on perfect squares, the students are to find the perfect squares between 100 and 400. Then Larry draws a table on the blackboard with the numbers from 1 to 14 and their squares, and he asks the students what the relationships are between the numbers. After some consideration, one of the boys, Steve, says: "Between each number there is an increase of two". Larry does not ask for an elaboration, but, rightly I think, interprets Steve's comment to mean that the difference between two consecutive square numbers increases by two every time one moves one column right in the table. Larry writes the differences under the table (see figure 1), and asks, if the pattern continues, and if anybody has a suggestion for why this is so. The students claim that the pattern does continue, but unsurprisingly nobody explains why. Larry provides an explanation, but appears to realise himself that it is unintelligible. He does not pursue the question any further.

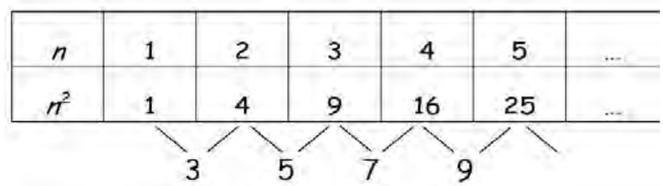

Figure 1. The beginning on the table on the blackboard.

In this episode Larry invites the students to engage in an activity that resembles what Cobb and his colleagues call reflective discourse (Cobb, Boufi, McClain, & Whitenack, 1997). Reflective discourse is a mode of classroom communication that involves a discursive shift, which makes a symbolisation of the results of a task the object of further investigation. In the episode above, the class uses the table as a starting point for finding patterns in the numbers. Such shifts are recurrent in Larry's classroom, but in the situation above, as well as in many others, classroom communication stops short of making further investigations of the conjectures and of justifying them. The students, then, engage only in a single run of the first two of the three elements of the reasoning and proof cycle (cf. the section 3), the ones concerned with initial exploration and making an initial conjecture.

Larry's reaction in this episode indicates that he engages sufficiently with mathematics in this situation to realise that Steve's observation does not qualify as a mathematical argument. He does not, however, ask Steve or other students for an elaboration, and he does not come up with an intelligible explanation himself. This is so although the geometric representations lend themselves to a generic argument, in this case an action proof (Reid & Knipping, 2010), for the conjecture: placing representations of two consecutive perfect squares on top of each other shows that the difference between them is the sum of the base of the two squares (e.g. that $5^2 - 4^2 = 5 + 4$; see fig. 2) and consequently that the difference increases by 2, when the base of the squares increases by 1.

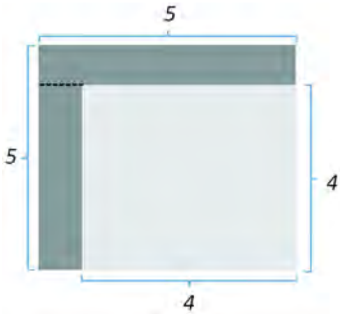

Figure 2. A basis for a generic proof: $n^2–(n–1)^2 = n+(n–1)$ and $((n+1)^2–n^2)–(n^2–(n–1)^2) = 2$.

Discussion of the episode

Larry was not asked to comment on the episode in an interview, and there may be many different explanations for what happened. I do not suggest that he should necessarily have pursued Steve's conjecture; much more needs to be known about the situation to warrant such a suggestion. However for R&P to play a role, the teacher must from time to time engage the students in further work on such conjectures. The discussion below addresses the question of what it would take for a teacher to capitalise on the R&P potentials of the situation, if this is deemed feasible in view of the broader context of the situation.

A TM interpretation of the situation (cf. the section 2) suggests that Larry has insufficient content knowledge to prove the conjecture, specifically that he lacks knowledge of the syntactic structures of mathematics. Another possibility, using the same framework, is that he knows how to represent perfect squares with cubes, but not how to use the representations to build a generic argument about the differences between perfect squares. In this interpretation the part of his PCK that links the contents to teaching is limited.

These are viable understandings, but PoP offers an interpretation that focuses less on Larry's (lack of) knowledge and more on how he draws (or not) on social practices beyond the classroom, as he engages in the ones that unfold within it. From this perspective Larry does not in this situation draw on the mathematical practice of *proving why*, specifically on the one of *proving why* by substantiating a number theoretical argument with geometrical means. One likely reason is that his previous experiences with doing so are insufficient for him to reengage in such a practice in the classroom. Another possibility is that Larry at the instant takes the attitude of the school in general and is primarily concerned with its traditional testing schemes and therefore reluctant to spend more time on issues that are bound not to come up in the test (e.g. proving). A third possibility is that he sees himself through the eyes of his students and seeks to avoid the resistance that may arise, if he pushes for what from their perspective is an unneeded, non-empirical argument.

The PoP interpretations above are somewhat speculative, as Larry did not comment on the episode. The main point, however, is that rather than viewing teaching as enactments of reified mental constructs (teachers' knowledge and beliefs), teachers' contributions to classroom practice may be viewed as linked to and transformed by their reengagement in a range of other practices that mutually transform one another in the process. As seen from this perspective, Larry's involvement with the reform discourse and its emphasis on mathematical reasoning is challenged and transformed, as he negotiates the significance of the more traditional assessment strategies and educational discourses at Mellemvang.

6. R&P in mathematics teacher education

The episode from Larry's classroom indicates that a teacher needs comprehensive prior experiences with *proving why*, if the reflective discourse is to be taken beyond the mere observation that Steve makes and focus on how the conjecture may be substantiated. Such a move may provide the students with opportunities to develop a better understanding of the contents of the claim as well as a growing sense of what a mathematical argument is.

From a PoP perspective, however, there is more to be said about this. To make my point I refer to the continued work on perfect squares in Larry's classroom. In the following lesson, the students solve tasks in the textbook, calculations of the type '$9^2 - 7^2$', '$8^2 + 6^2$', and '$12^2 - 6^2$'. As the students finish, Larry asks them to make more tasks for themselves and for their partner at the table. Two girls, apparently seeking an easy way out, make subtraction tasks in which the base of the subtrahend is 1: $9^2 - 1^2 = 80$; $3^2 - 1^2 = 8$; $7^2 - 1^2 = 48$. Without giving it much attention, they comment that the results are all in the 8-times table.

The two girls never raise the issue in the subsequent whole-class discussion, and I do not know, how Larry would have reacted, had he known that they were close to conjecturing that for any odd natural number, n, $n^2 - 1$ is divisible by 8. The question I address, however, is a more general one: what prior experiences with R&P do teachers need, if they are to make reasoned decisions on whether to pursue such an issue and, if they decide to do so, to engage their students in examination of the conjecture?

From a PoP perspective, the question is what it takes to make mathematical reasoning one of the possible practices that the teacher may draw on as (s)he interacts with the students?

As mentioned before, there is little reason to expect that a high academic level of mathematics in teacher education in and by itself improves instructional quality and student learning in schools. Merely exposing prospective teachers to large numbers *proofs that* in abstract algebra or calculus is unlikely to contribute significantly to how they handle R&P in elementary school. The practices of a university course in pure mathematics are too distant from the ones related to the multidimensional complexities of instruction for the former to significantly inform the latter. In classroom interaction teachers may seek to facilitate their students' learning; create an atmosphere in which students feel safe; position themselves among their colleagues; reengage in broader educational discourses; and much more. Doing so they take the attitude of their students, their colleagues, the parents, and others and reengage in practices and discourses linked to the respective communities. But the meaning they make of classroom interaction is unlikely to be informed by academic mathematics, even when teachers try to support their students' learning.

The moral of this is that the practices of teacher education programs need to fulfil two requirements. First, they are to be 'sufficiently close' to school mathematics to be drawn upon as the teachers engage with their students in their practicum and upon their graduation. This requires that the contents and tasks prospective teachers work with relate in a fairly immediate sense to instruction and student learning in schools. Second, teacher education practices need to be 'sufficiently close' to the academic mathematical practice of *proving that*, as prospective teachers need to reengage in R&P for verification purposes, when they assess the quality and relevance of a student's conjecture and decide if and how to pursue it in the classroom. If the teacher does not know how to prove if $n^2 - 1$ is divisible by 8, (s)he is unlikely take up the conjecture in the classroom; and if (s)he does so anyway, the students are unlikely to appreciate that one advantage of R&P-processes is that they provide us with better products in the form of a relatively more secure basis for our claims.

The proposed development project seeks to balance *proving why* with *proving that* by using conjectures that students make in investigative classrooms as starting points for R&P in mathematics teacher education. This could be a conjecture like *if n is odd, 8 divides* $n^2 - 1$. Prospective teachers will be asked (1) to develop different ways to prove *that* conjectures are right or wrong; (2) to consider if a proof also proves *why* the conjecture is right, and if it does not, try to find others that do; (3) to consider how a teacher might react if (s)he wants to pursue an agenda of mathematical reasoning in the situation at hand; (4) to use their own solutions as starting points for generating new conjectures to investigate. The research part of the project is to understand if and how the development initiatives provide prospective teachers with a better background for supporting their students' activities with R&P.

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Glimpses of students' mathematical creativity, which occurred during a study of students' strategies for problem solving in upper secondary mathematics classes

Abstract. This article reports the first results of a research project focusing on the development of the students' inquiry, creativity and intellectual independence when working in a problem solving setting in upper secondary mathematics classes. Eight mathematics teachers prepared and conducted teaching experiments for our observation of the new strategies gradually developed by the students. The theoretical basis of the project includes works of Schoenfeld, Polya and Cobb et al., Johan Lithner and, in the later parts, Andrea A. diSessa. The research project, being part of the Lifelong Learning Programme project KeyCoMath, studied the students' strategies for problem solving in mathematics. The article presents a didactic concept that emerged during the study: a particular aspect of the students' work, which we interpret as glimpses of mathematical creativity (GMC). Possible connections between GMC and metarepresentational competence are discussed.

Introduction

This article consists of two main parts followed by a common discussion and conclusion:

Part 1 contains the background and foundation of a research project, running with the aim to develop and study teaching that encourages students' activity, inquiry and autonomy, and corresponding goal directed mathematics learning. The project is part of the EU project KeyCoMath

Key words and phrases: Inquiry and creativity, problem solving, upper secondary students' strategies, metarepresentational competence, KeyCoMath.

AMS (2000) Subject Classification: Primary 97C99, Secondary 97D20.

(http://www.keycomath.eu/). It started in the spring 2013 where a collaborative research group was formed consisting of eight mathematics teachers from five local, upper secondary schools and one university researcher in mathematics education (the author of this article). This group had articulated certain concerns like:

i) Students are too dependent of check lists and working habits; seldom are they able to 'think outside the box'

ii) Even the brightest students can reproduce, but rarely produce mathematical thinking and

iii) Many students do not want to solve new problems or to answer new questions.

The group had the hypothesis, that appropriate problem-solving environments could support realization of many students' hidden potentials for independent, mathematical thinking. Therefore, we formulated the research question: "What strategies can we identify when the students work in an inquiry based learning environment in upper secondary mathematics?" During the first year of the project, the teachers had designed and taught sequences in their own classes of about ten lessons each, where the students worked with problem solving. Gradually, our group's research interest concentrated on students' modes of reasoning and, in particular, on ideas about mathematical creativity presented by Lithner (2008).

The teachers deliberately designed sequences, which should provoke examples of mathematical creativity. During the following years, we studied these teaching experiments and analyzed data with the aim to study episodes of mathematical creativity. We designed and taught new teaching experiments based on experiences from the first and second round. Analysis of these latest data is still going on.

Part 2 presents one of the main results of the research project in the form of a didactic concept, which emerged during the teaching experiments: a particular aspect of the students' work, which we interpret as glimpses of mathematical creativity (GMC). An episode of GMC, picked out from data from one of the teaching experiments, serves to illustrate how this concept emerged from our interpretation of the students' activities. In this case, the GMC occurred when pairs of students worked together in the experimental learning environment. Characteristic of the environment was the demand that the students engaged in solving a mathematical problem which was completely new to them, and also that the teacher deliberately would avoid to interfere by, for example, asking the students sub questions or structuring their actual process of problem solving. After minutes of work and unsuccessful trials, one of the students suddenly saw a solution in a glimpse. He explained it to the other student who immediately accepted the solution. Subsequent data analysis did not reveal any connection between this solution and the two students' proceeding ideas or suggestions. The solution diverged from the solution and the learning trajectory envisioned by the teacher when he designed the teaching experiment.

Analysis and discussion of GMC and the episode take place in the article's common final section, based on the research project's theoretical framework. The main issues for discussion are: "Can we interpret GMC as one type of Creative Mathematically founded Reasoning (CMR) (Lithner 2008)?" and "What are the connections between GMC and Meta-Representational Competence (MRC) (DiSessa 2002)?"

1. The research project

The research group's work was based on Polya's problem solving heuristics (Polya, 1985), Alan Schoenfeld's theories about mathematical thinking and problem solving and Johan Lithner's research into students' strategies for solving tasks and problems in mathematics (Lithner 2008). The research methodology was in line with a sociocultural perspective and encompassed collaborative teaching experiments (Cobb 1999). Data interpretation and analysis took norms and beliefs as its starting point and included social and psychological perspectives (Yackel and Rasmussen 2002).

During the project, we combined the teachers' designs of materials, and their teaching experiments, with discussions and exchange of experience in a number of joint meetings. The author collected data from the first round, in the form of video recordings, notes and materials. The data collection took place in 2013 – 2014 under the teaching experiments and the joint meetings. When the idea of GMC emerged during the analysis of data, the concept of metarepresentational competence (diSessa 2002, 2004) was included as part of our theoretical framework.

1.1. Polya and Schoenfeld: how to solve it and what it takes to solve it

A central theoretical contribution of Alan Schoenfeld's problem-solving research was his framework for analysis of mathematical problem-solving behavior. Based on discussions in the research group of (Schoenfeld, 2011), we decided to devide the teaching experiments into two separate parts. In every classroom experiment, the first session contained an introduction to an inquiry, problem solving working style. The other part was the main problem solving session. We planned to let the teaching of mathematical problem solving include explicit use of Polya's scheme (Polya, 1985).

The teachers did not in advance see this as a major change in their classroom practices because they felt that problem-solving strategies would also be taught normally, although implicitly. However, they had the general impression that their students were in need for elementary problem solving tools like, for example, those strategies based on Polya's scheme. The teachers wanted to enable the students to make progress on their own hand rather than call for help as soon as they felt lost. In particular, some of the teachers also wanted to get rid of the students' very close use of the textbook's list of answers to the tasks. The experiment aimed to widen the students' picture of mathematics in the direction of a subject open for ideas and including discussions based on mathematical knowledge and imaginations. The teachers wanted to change the students' beliefs about mathematics and about their own roles, and the project intended to contribute to a change of the classroom's norms and practices.

The teachers felt comfortable with the preparation of materials for both parts of the teaching experiment, supported by discussions in the group and in smaller meetings.

1.2. Lithner: Types of reasoning for solving tasks

According to (Lithner, 2008), solving a task can be seen as carrying out four steps:

1. A (sub) task is met, which is denoted problematic situation if it is not obvious how to proceed.
2. A strategy choice is made. It can be supported by predictive argumentation: Why will the strategy solve the task?
3. The strategy is implemented, which can be supported by verificative argumentation: Why did the strategy solve the task?
4. A conclusion is obtained.

Further, Lithner discerns between different types of reasoning involving strategy choice and strategy implementation. The two main types of reasoning are IR (Imitative Reasoning) and CMR (Creative Mathematically founded Reasoning). IR encompasses i) memorised reasoning where the strategy choice is founded on recalling a complete answer and the strategy implementation consists only of writing it down, and ii) three subtypes of algorithmic reasoning where the strategy choice is to recall a solution algorithm without creating a new solution; hereafter, the remaining parts of the strategy implementation are trivial.

In contrast, CMR fulfils all of the following criteria (Lithner, 2008, p. 266):

1. Novelty. A new (to the reasoner) reasoning sequence is created, or a forgotten one is re-created.
2. Plausability. There are arguments supporting the strategy choice and/or strategy implementation motivating why the conclusions are true or plausible.
3. Mathematical foundation. The arguments are anchored in intrinsic mathematical properties of the components involved in the reasoning.

Lithner did his studies at undergraduate level. Our group decided to take students' CMR as a goal for the teaching experiment. Therefore, the data analysis concentrated on the identification of episodes of students' creative mathematical thinking.

1.3. DiSessa: metarepresentational competence

Metarepresentational competence (MRC) refers to the full complex of abilities to deal with representational issues. It includes, centrally, the ability to design new representations, including both creating representations and judging their adequacy for particular purposes. But it also includes understanding how presentations work, how to work presentations for different purposes and, indeed, what the purposes of representations are. Knowledge that allows students to learn new representations quickly and the ability to explain representations and their properties is also included (diSessa 2002). Representational literacy is important for the students' critical capabilities (meaning the capability of judging the effectiveness of the design's result, and of redesigning it) in MRC, according to (diSessa 2002). According to diSessa (2004), MRC may account for some parts of the competence to learn new concepts and to solve novel problems. Our group's observations were in line with this and gave inspiration to new inquiries. Further, diSessa (2004) suggests that because insight and competence often involve coming up with an appropriate representation, learning may implicate developing one's own personally effective representations for dealing with a conceptual domain.

Although these two studies (diSessa 2002, 2004), in contrast with our project, aim at linking metarepresentational competence with design, and with students' critical capabilities, we found the concept of metarepresentational competence potentially useful for analysis of the GMC's occurring from our data.

2. Glimpses of mathematical creativity

2.1. Teaching polygonal numbers

This episode took place in a highest-level mathematics classroom with about 22 students. The teacher gave an introduction of Polygonal numbers, based on his oral explanation of how the next polygonal number emerged from the previous by expanding the polygon, and based on his drawings on the blackboard (Figure 1).

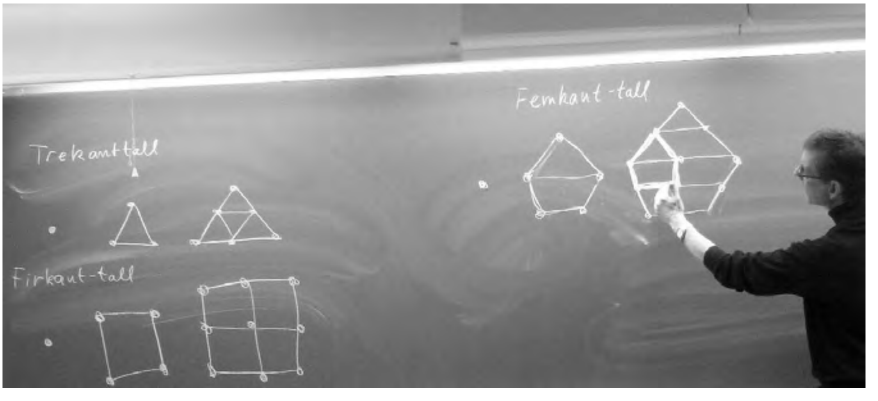
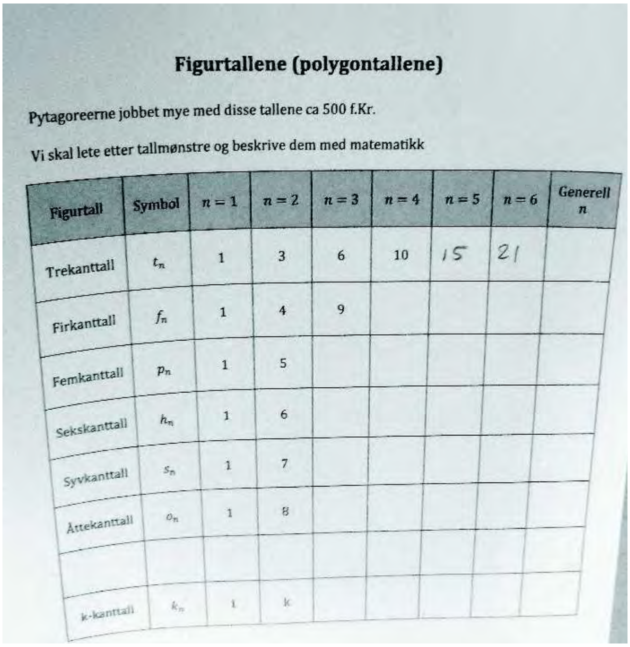


Figure 1. The teacher's introduction of polygonal numbers.

The students' task was to complete a form, distributed by the teacher, with the polygonal numbers and to express the general terms (Figure 2). After the introduction, the students started to work in pairs. The subject polygonal numbers was new to the students and they had no prior experiences (from the classroom, according to the teacher) with thhis kind.

There was no restrictions on what methods they might use, but the teacher gave no hints or sub-questions, neither. One strategy for completing the form would be to study the pattern of increase, as shown in Figure 3.

Most of the students combined drawings with counting and, simultaneously, looked for patterns in the rows and/or columns containing the numbers obtained from the drawings.



Figurtallene (polygontallene)

Pytagoreerne jobbet mye med disse tallene ca 500 f.Kr.

Vi skal lete etter tallmønstre og beskrive dem med matematikk

Figurtall	Symbol	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$	Generell n
Trekanttall	t_n	1	3	6	10	15	21	
Firkanttall	f_n	1	4	9				
Femkanttall	p_n	1	5					
Sekskanttall	h_n	1	6					
Syvkanttall	s_n	1	7					
Åttekanttall	o_n	1	8					
k-kanttall	k_n	1	k					

Figure 2. The form.

		n=1	n=2	n=3	n=4	n=5	n=6	General
Triangular Numbers		1	3	6	10	15	21	$\frac{1}{2} \cdot n(n+1)$
	Increased by	2	3	4	5	6	n	
Square numbers		1	4	9	16	25	36	n^2
	Increased by	2 · 2 − 1	2 · 3 − 1	2 · 4 − 1	2 · 5 − 1	2 · 6 − 1	2 · n − 1	
Pentagonal numbers		1	5	12	22	35	51	$3/2\, n^2 - \frac{1}{2}n$
	Increased by	3 · 2 − 2	3 · 3 − 2	3 · 4 − 2	3 · 5 − 2	3 · 6 − 2	3 · n − 2	
Hexagonal numbers		1	6	15	28	45	66	$2n^2 - n$
	Increased by	4 · 2 − 3	4 · 3 − 3	4 · 4 − 3	4 · 5 − 3	4 · 6 − 3	4 · n − 3	
k-polygonal numbers		1	k	3(k − 2) − (k − 3) + k	...	...	...	$\frac{1}{2}k(n^2 - n) - n^2 + 2n$
	Increased by		3(k − 2) − (k − 3)	4(k − 2) − (k − 3)	5(k − 2) − (k − 3)	6(k − 2) − (k − 3)	n(k − 2) − (k − 3)	

Figure 3. One strategy for completing the form.

2.2. The episode

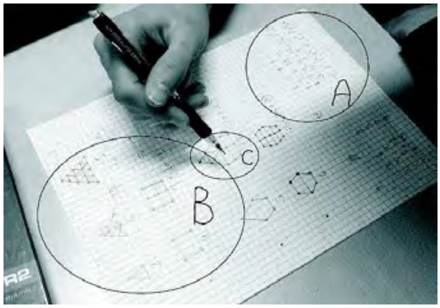
<table>
<tr><td></td><td colspan="2">Danielsen 18. 03. 2014 video0008, 00:02:02 – 00:04:18.</td></tr>
<tr><td>1</td><td>The two students B1 and B2 sit and work together. They have already managed to write the first five triangular numbers 1, 3, 6, 10 and 15 and the square numbers 1, 4, 9, 16, 25, 36, 7^2, 8^2 and 9^2 (Area A, Figure a and b) based on their drawings (Area B, Figure a).
Apparently, their unarticulated plan was to find a pattern for the extension from triangular numbers to square numbers, which they could extend to create the pentagular numbers and, afterwards, the succeeding polygonal numbers.
B1: ... Then the next one is seven squared, (writes 7^2), the next one is eight squared (writes 8^2), the next is nine squared (writes 9^2 in Area A, Figure a and b)</td><td>

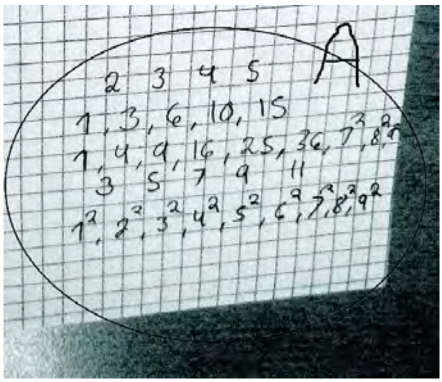
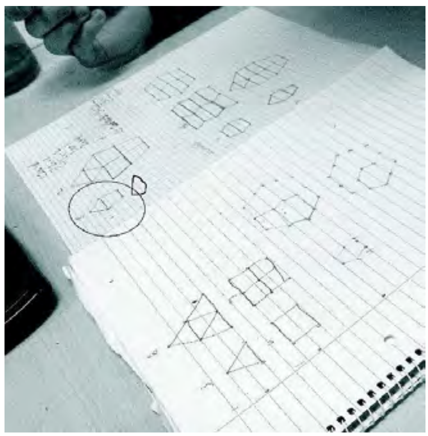
Figure a


Figure b</td></tr>
<tr><td>2</td><td>B1: then we know the difference between these (points to the square numbers, points to the numbers 3, 5, 7, 9, 11 in area A Figure a and b)</td><td>Their preliminary choice of a strategy was, apparently, to read a pattern from the increase of the square numbers. The teacher's introduction had lead them in this direction (without giving any details, though)</td></tr>
<tr><td>3</td><td>B1: ... so in fact you have (writes 1^2, 2^2, 3^2, 4^2, 5^2, 6^2, 7^2, 82, 9^2, last line in area A, Figure a and b)</td><td>B1 rewrites the square numbers in powers of two, apparently for making it easier to read a pattern</td></tr>
</table>

4	B2: *But we cannot...* B1: *How can we write a formula for this?* B2: *For the triangle, it is not squared at least* B1: *But the triangle is different* (...)	It becomes clear to B1 and B2 that the pattern they look for cannot be as simple as an increase in powers
5	B1: *The triangle, it is something with its three sides, with the triangle in the middle somehow...* (points to area C, Figure a) B1: (draws a triangle, covered by his hand on Figure a)	30 seconds silence B1 and B2 are both starring at their drawings. According to my interpretation, they reconsider the strategy and try to take inspiration for a new strategy
6	B1: (...) *one more. What is the formula for the square?* B2: *Yes I see that, the square is okay. But..*	This sounds as if B1 still considers the old strategy of extension, and maybe he wants to check it out again. B2 is finished with the squares and he does not reconsider the same extension idea
7	B2: *But then, the triangle, you can somehow...* (points to the polygon in area D on his drawing Figure c)	 **Figure c**
8	B2: *For example, for the pentagonal, then you may in a way, you can take a formula for the triangle and a formula for the square and add them*	B2 takes inspiration from his drawing to express the pentagonal numbers by a formula, which he can create by adding the formulas they already know.

9	B1: *Then you can use it for all* B2: *Yes you can do it with all of them* B1: *Yes, exactly. So This is the square* (points to the squares in area B in Figure a), *that is why it becomes like this...*	B1 acknowledges that the principle is applicable for all the pentagonal numbers and B2 agrees. B1 recognises the squares as parts of the pentagonal numbers in the first few cases in his own drawings
10	B1: *For example these points here, they have two in common...* B2: *Yes yes...*	They start to figure the formula out as a sum when taking into account that the triangle and the square has one line in common, according to the drawing

Figure 4. The episode.

2.3. Problem solving and mathematical creativity

This episode may serve to give insight into interplay between problem solving and mathematical creativity. The two students did neither perceive, nor try to solve the problem as a routine task, but they engaged in autonomous and independent thinking. This was what we were aiming at in the project.

The episode was typical with regard to the lack of discussion of, or negotiation about the students' strategy choice (step 2 in Lithner's task solving, above). Instead of that, each of the two students chose his strategy and 'thought loud' while the other commented on and gave response to his ideas, and vice versa. None of them seemed to expect that they would come to any agreement about the strategy. One of the students saw a convincing solution strategy in a glimpse, shared the idea with the other student and immediately, they both started to implement the strategy although it represented a novel idea to them.

3. Analysis and discussion

Can we interpret GMC as one type of Creative Mathematically founded Reasoning (CMR)? The four criteria were, in italics (Lithner 2008):

1. *Novelty. A new (to the reasoner) reasoning sequence is created, or a forgotten one is re-created*: In the case, B1 and B2 both experienced new insight and established a new relation between figure and formula.

2. *Plausibility. There are arguments supporting the strategy choice and /or strategy implementation, motivating why the conclusions are true or plausible*: Immediately after the GMC in the case, B1 and B2 started to verify their new idea by calculations and arguments.
3. *Mathematical foundation. The arguments are anchored in intrinsic mathematical properties of the components involved in the reasoning:* In the case, this could be an issue of discussion. The arguments in the case were anchored in the change of representation.
4. *CMR does not, as problem solving, have to be a challenge. The definition also includes elementary reasoning.* The students in the case got new insight, which could be interpreted as a solution to a problem as well as elementary reasoning depending on the definition of elementary reasoning.

The criteria for CMR were fulfilled in the case, only if shifts between different representations can be seen as part of the 'mathematical foundation' in bullet 3. The GMC in the episode was founded on the students' competence in shifting between the different representations (numbers, formulas and drawings) of the polygonal numbers. Their arguments for supporting the strategy choice (plausibility, bullet 2.) were anchored in both students' representational literacy, which is an aspect of metarepresentational competence (MRC) as it was described above.

One of the students' representational literacy was revealed in the episode's scheme row 8, where B2 talked about the formula for the triangle and for the square, and about adding these two, without even to discern between the different representations. Neither did he count the points of each pentagonal, one after the other, nor add the corresponding triangular numbers and square numbers. He simply handled the problem by identifying the figures with the formulas. The other student immediately understood the idea.

This episode illustrates how the experimental lesson on polygonal numbers, founded on interplay between different representations, could be supportive of the students' development of metarepresentational competence as well as their creative, mathematically founded reasoning. The student's creative reasoning in the episode happened in a glimpse when he caught the connection between the pentagon consisting of a triangle (with a corresponding formula) and a square (with a corresponding formula) on the one hand, and, on the other hand, the algebraic number of which he wanted to have a formula. Meaning that the GMC happened in the moment, when the student managed to see the two as different representations of the same object. A number of episodes from our data contains examples of GMC, which happen in a similar way when a student manage to establish a link between two different representations of the same object. Further analysis of data from our group's experiments may provide interesting insight into the connections and relations between CMR, GMC and MRC.

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The effects of heuristic strategies on solving of problems in mathematics

Abstract. It is a universally accepted truth that problem solving forms the basis for successful mathematics education. Problem solving is an indicator of the state of comprehension of the concepts that pupils are taught. They help their solvers realize what former knowledge is applicable in a new situation, what role this knowledge plays in it, and which piece of knowledge turns out to be useless, or even erroneous, and thus becomes an obstacle to further development of mathematical knowledge and pupils' skills.

The text presents the results of a three-year project, Development of a culture of solving mathematical problems in Czech schools (Czech Science Foundation project P407/12/1939) focusing on the use of heuristic strategies in problem solving. Heuristic strategies have been used in Polya's and Schoenfeld's understanding of the concept. The theoretical background of the research was Brousseau's Theory of Didactical Situations.

The use of heuristic strategies will be explored from two different perspectives: how heuristic strategies develop pupils' understanding of mathematics through using them, and how teachers change in consequence to giving their pupils the chance to use these strategies.

Key words and phrases: problem solving, heuristic strategies, culture of problem solving, creativity.

AMS (2000) Subject Classification: 97B10.

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1. Introduction

Problem solving forms the basis for successful mathematics education. Solution of carefully selected problems helps to develop, refine and cultivate creativity (Kopka, 2010). The situation in which mathematics is taught as mere transfer of formulas and ready-made algorithms etc. from the teachers to their pupils results in pupils' believe that "an average pupil stands no chance to discover any useful idea... If we want to show pupils what mathematics is, it is best to solve problems with them". (Kopka, 2010, p. 13) If the teacher chooses problems in whose case the search for the appropriate algorithm is easy and also often hint at the suitable solving procedure are explicit, then pupils instead of solving a problem simply apply some algorithm chosen according to the signals from the assignment or the teacher. They fail if they are to solve non-standard problems whose assignment does not contain elements they are used to or elements that serve as indicators for selection of the right solving strategy. They feel helpless if they face an atypical, unusual problem or a problem set in an unknown context (it often happens when pupils are expected to use mathematics when solving problems from everyday life).

It is quite easy to change the plan of a sequence of lessons (the taught concepts, their properties, organizational forms etc.). What is much more difficult is to choose suitable problems for teaching because in doing so both mathematical and didactical perspectives must be respected. At the same time problem solving is an indicator of the state of comprehension of concepts that pupils are taught. They help their solvers realize what former knowledge is applicable in the new situation, what role this knowledge plays in it, what knowledge turns out to be useless or even erroneous and becomes an obstacle to further development of mathematical knowledge and pupils' skills. Problems should develop pupils' intellectual activity, simulate work of a mathematician who is facing a problem to be solved, encourage solvers' creativity in the solving process (Brousseau & Novotná, 2008). If problems are to meet these criteria, it is not enough to look for good assignments (although we cannot do without them). It is essential to create stimulating learning environments that influence pupils' relationship to problem solving.

It is a generally accepted fact that the ability to solve problems develops fast if the solver gains new experience in this activity. Pupils' performance in problem solving improves if they repeatedly meet problems of similar type in case they can make use of their past experience (Eysenck, 1993). The case when the "quality" of solving improves when past experience is used is called *positive transfer.*

Observations from Czech schools confirm that not only pupils, but also teachers prefer problems where there are no doubts about the appropriate algorithm. The pupil then does not have to take the painstaking and difficult journey to discovery of the appropriate algorithm and to grasping of the problem. The teacher's role is reduced to mere detection of the places where his/her pupils made mistakes and to assessment of the correctness of their solutions. That is why teachers often select problems in which case the discovery of the appropriate algorithm is straightforward and provide their pupils with clues on the solving procedure. In consequence, instead of solving a problem, pupils merely apply an algorithm selected according to signals from the problem assignment. It is no surprise that so many pupils then fail while solving non-standard problems whose assignment does not include what they are used to and what they are usually guided by in the solving process. They feel at a loss if assigned an atypical, unusual problem or a problem from an unknown context. Researches confirm that these difficulties are much less frequent in case of pupils whose teachers often assign them e.g. word problems of non-algorithmic nature (Eisenmann, Novotná & Přibyl, 2014).

Any changes in approaches to problem solving in school practice are conditioned by changes in teachers' attitudes to mathematics education at schools, see e.g. (Tichá & Hošpesová, 2006). Mathematics education based on problem solving with no transfer of ready-made knowledge to pupils, i.e. on creative solving, must be built on thorough teachers' knowledge of mathematics, on their own experience with creative approach to problem solving, but also on easy access to information and knowledge ready to be used in teaching. Equally important is also the so called *specialized content knowledge* (Ball, Thames & Phelps, 2008). This knowledge involves identification of key mathematical concepts and of the potential this activity bears, detection of various forms of representation of mathematical concepts and operations, including their advantages and drawbacks.

Changes of pupils' attitude to problem solving are one of the phenomena studied in the frame of the GAČR research project *Development of* *culture of problem solving in mathematics in Czech schools.* The project explored the possible ways of changing pupils' attitudes to problem solving, of making pupils aware that mathematics problems are the means needed for their own personal development. One of the key research questions was to what extent this approach has positive impact on development of pupils' understanding, on their attitudes to creative approach to problem solving and on their reactions when meeting a modified or brand new assignment, on their coming up with new, original solving procedures when solving a new problem. These are perceived as indicators telling the teacher whether a pupil understands the subject matter. In the project, the tools for development of the pupils' "culture of problem solving", for ways for changing pupils to "experts" in problems and problem solving strategies were created.

2. Theoretical background and methods

The Theory of Didactical Situations in Mathematics (Brousseau, 1997) states that for each problem there is a set of knowledge that enables its solution. However, when solving a problem not all the needed knowledge is necessarily available to the pupil. This means that learning is the process of broadening the repertoire of tools available to a pupil. The role of the teacher is to create such environment that supports this broadening. The teaching/learning process can be characterized as a sequence of situations (natural or didactical) that result in modifications in the students' behaviour that are typical for getting new knowledge (Brousseau, 1997).

In our research we focus on selected heuristic strategies of solving problems. These strategies are presented later. We developed a tool for description of a pupil's ability to solve problems. This tool is the structure Culture of problem solving (CPS) and is described in the following subsection.

2.1. Culture of problem solving

Culture of problem solving can be explored from three perspectives. The first group is formed by works that focus on description of pupils' attitude to problems and problem solving in dependence on different variables influencing these attitudes (Nesher, Hershkovitz & Novotná, 2003).

The second group is formed by works whose goal is to bring about change in the culture of problem solving both in case of an individual and of groups of pupils, and build-up of pupils' motivation to problem solving (Bureš & Hrabáková, 2008; Bureš & Nováková, 2010; Bureš, Novotná & Tichá, 2009; Bureš, Nováková, Novotná, 2010). The third group consists of works focusing on complex projects in problem solving, such as clusters of problems (Kopka, 2010; Bureš, 2010), mathematics rallies (Brousseau, 2001; Novotná, 2009; Růžičková & Novotná, 2010). In all cases, pupils work with sets of problems, solve them individually and in groups and then share their experience and knowledge from the solving process and discuss it.

The approach used in the here reported research conceives CPS as the tool for description of pupils' solving profiles. It allows us to measure the changes in pupils' attitude to problem solving, in their success rate and in the solving strategies they use. The structure of CPS is presented in detail in (Eisenmann, Novotná & Přibyl, 2014).

CPS used in the project consists of the following four components: intelligence, creativity, reading with comprehension and ability to use the existing knowledge. The first three components are measured by standard psychological tools and were assessed by a psychologist, the test for assessment of the ability to use the existing knowledge was created by the project solving team.

Psychological screening was conducted using the following tools.

Pupils' *intelligence* was tested by the Váňa's intelligence test (Hrabal, 1975). This test was selected because of its verified correlation with pupils' school performance. It is particularly appropriate for the age 11 to 15, suitable for investigating the intellectual level of whole school classes, of the level of individuals' cognitive abilities (esp. of the component that conditions school success), in research situations where basic data about pupils are collected.

Pupils' *creativity* was investigated in the context of divergent thinking. Its level was measured using Christensen-Guilford test (Kline, 2000) that measures four dimensions: fluency (how many relevant uses the pupil proposes), originality (how unusual these uses are), flexibility (how many areas the answers refer to) and elaboration (quality and number of details in the answer).

Pupils' *ability to read with comprehension* is one of the key competences for successful problem solving. The pupils were presented with a short text (one paragraph) which they were asked to summarize in four lines without changing the meaning and content. The pupils' results are classified into the following categories: Comprehension of the meaning and preservation of all details, Comprehension of the meaning and preservation of substantial details, Grasping the meaning, more all less preserved content without details, Incomprehension of the original text and few details or wrong content, Incomprehension without presentation.

In the test of the *ability to use the existing knowledge* (AUK), pupils were assigned four pairs of problems, called *dyads* (Fig. 1). The first problem from the dyad tested the presence of certain knowledge, the second its use, e.g. in a non-algorithmic (non-standard) context. Each problem was evaluated by A (yes), i.e. the problem was solved, or N (no), i.e. the problem was not solved. Each dyad was evaluated as follows: AA (knowledge is present and used), AN (knowledge is present but not used), NA (knowledge absent but the second problem was somehow solved) and NN (knowledge not present and the second problem not solved). The best result is AA, the worst AN. The total evaluation of a pupil AUK was calculated by the formula 1 x AA + 3 x AN + 2 x NA + 2 x NN which expresses the pupil's success in the use of existing knowledge. It does not evaluate the school success.

a) State the volume of a cuboid with dimensions 3 cm $\times$ 5 cm $\times$ 100 cm.
b) The cross–section of a floor timber is a rectangle with the dimensions 15 cm and 25 cm, the length of the timber is 5 meters. How much will we pay for the timber, if 1 m^3 of wood costs 7 000 CZK?

Figure 1. Example of a dyad.

The tests used for determination of all four components of pupils' CPS were supplemented by assessments of pupils by their mathematics teacher based on interviews of the researchers with the teachers. In the interviews, attention was paid to surprising, unexpected pupils' results, both from the researchers' point of view and from the point of view of the teacher who had known the pupils for longer time and from various situations.

2.2. Heuristic strategies

The strategies we refer to as heuristics, in accordance with Polya (2004) and Schoenfeld (1992), are those solving strategies that pupils use to solve problems in another way than using school algorithms. Heuristic is the typical human way of solving problems.

The importance of heuristic strategies is discussed also by Vohradský et al. (2009). They emphasize that heuristic strategies allow the teacher to support independent solving activity based on inquiry and discovery, on the use of former knowledge, on posing of suitable questions etc. They point out that these strategies motivate pupils and help them grasp the content and master new knowledge but can never entirely replace other methods. If heuristic strategies are to be used successfully, it is "essential that pupils have mastered prerequisite knowledge and skills and that the goal they want to achieve be clear to them and adequate to their abilities. The main goal of heuristic strategies is development of independent, creative thinking in pupils." (Vohradský et al., 2009, p. 15).

The issue of the use of heuristic strategies suitable for lower and upper secondary school mathematics teaching is discussed, e.g. in (Novotná et al., 2013; Břehovský et al., 2013; Novotná et al., 2014a, 2014b). In (Bureš and Nováková, 2015) the method of implementation of heuristic strategies in class preceded by a detailed a priori analysis and followed by an a posteriori analysis and a comparison between the teacher's expectations and the reality in the class is studied.

Studied heuristic strategies

In the research, the following heuristic strategies were used; each strategy is illustrated by an example.

Guess – check – revise: First, based on our experience, we make a guess about the solution to the given problem. Then we check whether the solution meets the conditions of the assignment. The next guess is made with respect to the previous result. We carry on in this way until we find a solution.

Problem: Determine the two consecutive odd natural numbers so that their product is 323.

Solution: In Tab. 1 we gradually choose both consecutive odd numbers and investigate their product. With respect to the final column we make the decision whether to increase or decrease the numbers until we get the solution.

First odd number	Second odd number	Their product	Is the number 323?
1	3	3	No. This is (far) too little.
11	13	143	No. This is too little.
21	23	483	No. This is too much.
19	21	399	No. This is too much.
17	19	323	Yes. That's it.

Table 1. Recording of the solution.

Answer: The required numbers are numbers 17 and 19.

Systematic experimentation: Systematic experimentation is a strategy in which we try to find the solution to a problem using several experiments. First we apply some algorithm that we hope will help us solve the problem. Then we proceed in a systematic way and change the input values of the algorithm until we find the correct solution.

Problem: A bottle with a stopper costs 110 CZK. The bottle cost 100 CZK more than the stopper. How much is the stopper?

Solution:

The stopper costs 1 CZK. Then the cost of the bottle is 101 CZK. This does not correspond to the total sum 110 CZK.

The stopper costs 2 CZK. Then the cost of the bottle is 102 CZK. This does not correspond to the total sum 110 CZK.

The stopper costs 3 CZK. Then the cost of the bottle is 103 CZK. This does not correspond to the total sum 110 CZK.

The stopper costs 4 CZK. Then the cost of the bottle is 104 CZK. This does not correspond to the total sum 110 CZK.

The stopper costs 5 CZK. Then the cost of the bottle is 105 CZK. This is the correct solution.

Answer: The stopper costs 5 CZK.

Strategy of analogy: Analogy is a type of similitude. If we are to solve a particular problem we find an analogical problem, i.e. a problem that will deal with a similar problem in a similar way. If we manage to solve this similar problem, we can then apply the method of its solution or its result in the solution to the original problem. Solving a problem using analogy is very often a successful way to reaching the goal. It is characteristic of prominent mathematicians that they see analogy where nobody else is able to discern it. However, the method is tricky. Its use may also lead us to wrong conclusions. Not all properties of analogical objects must necessarily correspond. The pitfalls of the use of analogy in teaching mathematics are pointed at e.g. by Brousseau (1997) who speaks of "improper use of analogy". It is also discussed in (Novotná, Eisenmann & Přibyl, 2015).

Problem: Which fraction is greater: $\frac{125}{126}$ or $\frac{124}{125}$?

Analogical problem: *Which fraction is greater: $\frac{3}{4}$ or $\frac{2}{3}$?*

Here the answer is obvious: $\frac{3}{4} > \frac{2}{3}$.

Answer: $\frac{125}{126} > \frac{124}{125}$.

Problem reformulation: When using this strategy we reformulate the given problem and make another one which may either be brand new, is easier for us to solve and which solution is either directly the solution to the original problem or facilitates its solution. A specific and very important example of this strategy is translation of a word problem from one language of mathematics to another. Classical geometrical problems such as trisection of an angle were easy to solve when translated to the language of algebra.

Problem: Which fraction is greater: $\frac{125}{126}$ or $\frac{124}{125}$?

Reformulated problem: *We have two identical pizzas (or two congruent circles). We divide one of them into 125 identical pieces, the other to 126 identical pieces. We take away one piece from each pizza. In which pizza will there be more left?*

Solution to the reformulated problem: As we are dividing the same object, the pieces in the second one are smaller (we divide the same area into more identical parts). As we have taken away a smaller piece from the second pizza, more must be left there.

Answer: $\frac{125}{126} > \frac{124}{125}$.

Solution drawing: When using graphical representation we usually visualize the problem by making a drawing. We write down what is given and often also what we want to get. The drawing we get in this way is called an illustrative drawing as it illustrates the solved problem. Sometimes we can see the solution of the problem immediately in this drawing.

However, in most cases we must manipulate with the drawing (e.g., we add suitable auxiliary elements) and we solve the problem with the help of this modified drawing. We call this drawing the solution drawing.

Problem: We have a square inscribed in a circle and this circle is inscribed in another square. Determine which part of the larger square is covered by the smaller square (see Fig. 2).

Solution: Using a suitable rotation of the smaller square (see Fig. 3) and adding its diagonals enables us to solve the problem easily.

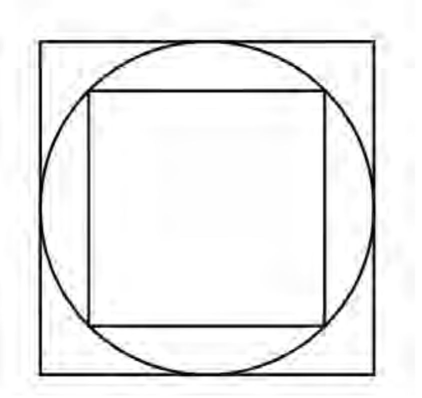

Figure 2. Picture illustrating the assignment.

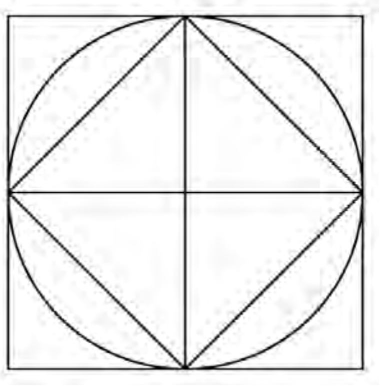

Figure 3. Rotation of the smaller square and introduction of an auxiliary element.

Answer: The smaller square occupies one half of the larger square.

Use of graphs of functions: When there are functions in the problem assignment or when it turns out within the solving process that it is desirable to introduce functions then it is usually good to draw graphs of these functions. These graphs often considerably contribute to finding the solution to the given problem.

Problem: Determine the number of roots of the equation $x^2 = 2^x$.

Solution: In Fig. 4, there are graphs of both functions x^2 and 2^x. The intersection points of the two curves represent the roots of he given equation.

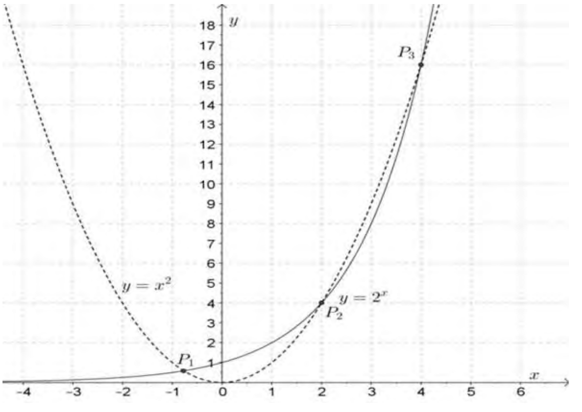

Figure 4. Graphs.

Answer: The equation has three roots.

Omitting a condition: Problem assignment often involves several conditions. If we are not able to fulfil all these conditions when solving the problem at once, we can ask: What is it that makes the solution of this problem so difficult? If we manage to identify which of the initial conditions is the difficult one, we can try to omit it. If we are then able to solve the simplified problem, we can go back to the omitted condition and try to finish solution of the original problem.

Problem: We have a classical chessboard with two opposite black corner squares removed (see Fig. 5). We have a sufficient number of domino blocks. It is possible to cover all the squares of this modified chessboard in such a way that none of the blocks sticks out of the board?

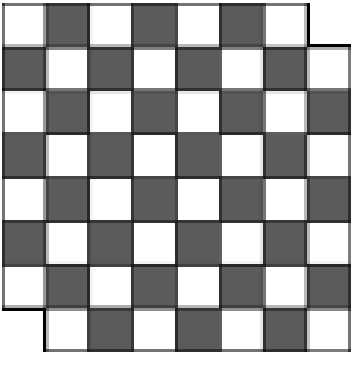

Figure 5. Chessboard with corners removed.

Solution: Let us omit the condition of the modified chessboard and let us work in a classical chessboard with 64 squares. It is relatively easy to cover this chessboard with domino blacks. Now let us accept the restriction – let us remove the opposite black corner squares from the solved problem (covered chessboard). Obviously, if we do this, two domino blocks are removed and two white squares remain uncovered. However, each domino block covers one black and one white square, therefore the remaining two fields cannot be covered by a domino block regardless of how we move the blocks on the board.

Answer: The modified chessboard cannot be covered by domino blocks.

Working backwards: This is a frequently used strategy in mathematics when we know the final state, the initial state and try to proceed "from the end to the beginning".The solution of the problem is based on "turning" the found solution round. It is often used in problems from the domain of geometric constructions.

Problem: One third of the whole number reduced by 20% equals 32. What is the number?

Solution: Let us suppose that we know the final number and are looking for the initial one. We will use the inverse operations:

– Instead of multiplying by $\frac{1}{3}$ we will multiply by 3.

– Instead of multiplying by $\frac{4}{5}$ (80%) we will multiply by $\frac{5}{4}$.

Answer: The number is 120.

Specification and generalisation: We choose a specific value or position, or we select a specific case, in the first stage. We solve the problem. If we can generalise the result of the problem, we formulate a hypothesis about the result of the original problem. We either leave the hypothesis on a plausible level, or prove it (if the solver's abilities are sufficient for it). If we cannot make the generalisation, we continue the solving process by another specification.

Problem: A shopkeeper bought a book at one seventh of the original price and sold it for three eighths of the original price. What was the shopkeeper's profit in percents?

Solution: Let us now specify the problem and let us presume that the original price was e.g. 56 CZK. This means the shopkeeper bought it for 8 CZK and sold it for 21 CZK. His profit is easy to calculate: $21 - 8 = 13$. The profit in per cents is: $\frac{13}{8} \times 100 = 162.5$.

Based on our specification we got one result. If we choose several different prices we can easily verify that the choice of the original price has no effect on the result. This allows us to generalise this result.

Answer: The shopkeeper's profit was 162.5%.

***Generalisation and specification*:** We choose a more general problem that we are able to solve. Using the specification we specify the answer for the original problem.

Problem: Which fraction is greater: $\frac{125}{126}$ or $\frac{124}{125}$?

Solution: Let us generalise the problem and let us formulate the following hypothesis:

$$\forall(n \in \mathbb{N}) \ \frac{n}{n+1} < \frac{n+1}{n+2}.$$

Proof of the generalised relation:

$$\begin{aligned} 0 &< 1, \\ n^2 + 2n + 0 &< n^2 + 2n + 1, \\ n(n+2) &< (n+1)^2, \\ \frac{n}{n+1} &< \frac{n+1}{n+2}. \end{aligned}$$

If we specify it for value $n = 124$ we get the solution of the given problem.

Answer:

$$\frac{125}{126} > \frac{124}{125}.$$

***Introduction of an auxiliary element*:** We try to transform a given problem to a problem we have already managed to solve, or we transform it into a simpler problem we are able to solve.

An example of an auxiliary element in geometrical problems is e.g. introduction of straight line or line segment, but it can also be a more complex geometrical figure. In algebra, we often introduce a new variable (substitution).

Problem: Square $CDEF$ is inscribed into an isosceles triangle ABC (see Fig. 6). What is the area of the square if line segment AB is 8 cm?

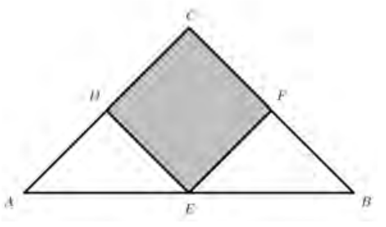

Figure 6. The given triangle.

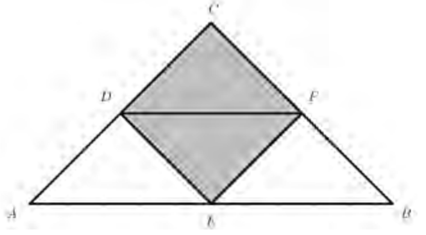

Figure 7. Introduction of an auxiliary element.

Solution: We have to realize that triangle ABC is formed by four identical triangles (ADE, EBF, DEF, CDF, see Fig. 7): Square $CDEF$ whose area is to be stated is made of triangles DEF and CDF, i.e. of two out of the four triangles that make triangle ABC, thus the area of square $CDEF$ will equal one half of the area of triangle ABC.

$$S = \frac{1}{2}\frac{|AB|\times|CE|}{2},$$

where $CE = 4$ cm, because CE and DF are diagonals of square $CDEF$; thus $|CE| = |DF|$ and moreover DF is a line joining the mid-points of AC and BC. Then we get $|CE| = |DF| = 4$ cm. After substitution $S = 8$ cm^2.

Answer: The area of square $CDEF$ is $S = 8$ cm^2.

***Decomposition into simpler cases*:** The problem is decomposed into some simpler cases that we are able to solve. The solution to the original problem is obtained by linking solutions to all simpler problems.

Problem: There is a parallelogram in Fig. 8. Draw two straight lines though vertex A which divide the parallelogram into three parts of equal area.

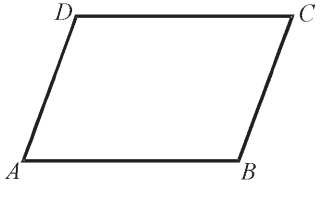

Figure 8. The given parallelogram.

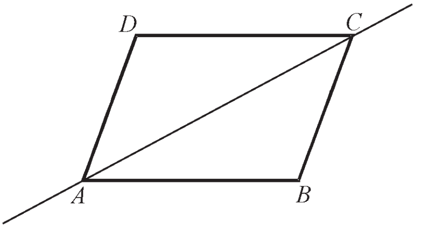

Figure 9. Creating triangles.

Solution: Let us divide the parallelogram by a diagonal from vertex A into two identical triangles.

Let us now consider the bottom triangle (see Fig. 9) and let us divide it into three identical parts. The new, simpler problem now is:

Draw two straight lines through vertex A of triangle ABC which divide the triangle into three parts of equal area.

Now it is enough to divide side BC by two points into three equal parts. Thus we get the vertices of the three required triangles (see Fig. 10). These three triangles have one identical side and height.

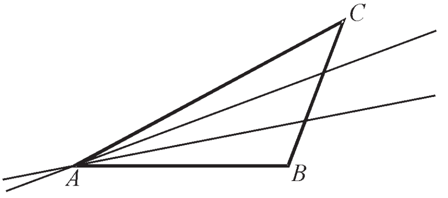

Figure 10. Division of the triangle.

Let us now go back to the original problem. If we divide the other triangle analogically, we get the following division of the parallelogram (see Fig. 11).

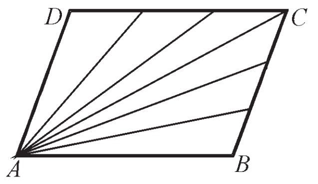

Figure 11. Back to the parallelogram.

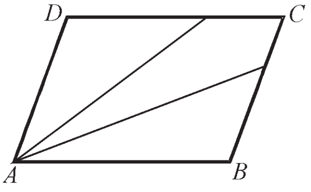

Figure 12. The final division.

The parallelogram is now divided into six parts of equal area. Thus the solution of the original problem is shown on Fig. 12.

***Use of false assumption*:** This strategy belongs to the family of experimental heuristic strategies. It can be well applied in problems where the value of a number in the problem is directly proportional to the result. The first value is selected with full awareness that the value is probably wrong (false assumption). The correctness of the estimate is verified. The assigned value is compared with the value calculated from the estimate and the proportion between them is found. The result is calculated using this finding. The mathematical background of this strategy is a linear function.

Problem: Perimeter of an oblong is 60 m. Determine the lengths of its sides if you know they are in the ratio 7 : 3.

Solution: To preserve the required ratio let $a = 7$ m, $b = 3$ m. Thus we get one half of the perimeter:

$$7 + 3 = 10$$

That is not enough. It must hold that $a + b = 30$. Therefore we must enlarge the lengths of both sides. How many times? Three times.

Answer: The length of one side is 9 m and the length of the other side is 21 m.

2.3. Methods

In (Novotná et al., 2013; Břehovský et al., 2013; Novotná et al., 2014a, b), focus is especially on introduction of heuristic strategies suitable for being used in primary and secondary schools when pupils solve problems using different methods than school algorithms. However, most pupils cannot be expected to start using these heuristic strategies unless they are given help, either from the teacher or somebody else from outside the school.

Within the project, two experiments were prepared and conducted – a short-term and long-term one. In the short-term, four month experiment, pupils were repeatedly introduced to the advantages of the use of selected heuristic strategies in solving mathematics problems. The main goal was to find out in case of which strategies short-term period is sufficient to teach pupils use these heuristic strategies. Our attention was also paid to whether the use of these strategies brings positive changes in pupils' attitude to solving of mathematics problems, improves their understanding of mathematics and develops their ability to use mathematics in different situations. (Novotná, Eisenmann & Přibyl, 2014a)

The long-term, sixteen month experiment focused on the study which heuristic strategies and how successfully pupils are able to learn to use actively and whether long-term use of heuristic strategies in problem solving affects the components of CPS.

The following two hypotheses were formulated at the beginning of the experiments:

1. At post-experiment pupils will be able to use some heuristic strategies in problem solving actively and to a greater extent.
2. At post-experiment pupils will have markedly better results in all components of CPS.

The organization of the teaching units in the classrooms was the same in both experiments: The teachers assigned a problem to their pupils. They let them work and asked the pupil who was the fastest to solve the problem correctly to explain their solution to the others. This was followed by a discussion and explanation of the solving strategy. The teacher then asked other successful solvers to present alternative solutions to the others. They were always asked to justify their solving procedures. If none of the pupils solved the problem with the intended heuristic strategy, it was demonstrated by the teacher. In another, similar problem the teacher then checked to what extent the teacher's solution was actively understood. The pupils were always encouraged to look for more ways of solving a particular problem and to record their problem-solving procedure. Sometimes the problems were set for homework. However, the way of working with the solutions at school was consistent.

The substantial difference between usual lessons the pupils were used to and the teaching experiments was the way in which problems were solved with the pupils. The teachers' working procedure in the lessons was the following: they presented the problem to their pupils (mostly in the written form, on a worksheet). They let them work and after some time (when at least one half of the pupils had solved the problem) they asked one pupil to present the solution to the others. Then they checked whether the rest of the class had understood the presented solution and invited the pupils to show their own solution if it was different. If the solution which was the aim of setting the problem did not appear among the presented solutions, the teacher demonstrated it to the pupils. The pupils were always encouraged to look for more ways of solving a particular problem and to record their problem-solving procedure. In the discussions they were asked to justify their procedures. Moreover, the teachers were teaching their pupils to recognize the used heuristic solving strategies and distinguish between them.

The participating teachers were provided with a sufficient number of problems that are solved most efficiently using one of the considered heuristic strategies. From this list of problems posed by the research team, the teachers were selecting the problems they found suitable for use in their classrooms.

Short-term experiment

The short-term, four month experiment was conducted in 11 classes (4 basic school classes with 12-year-old pupils, 4 basic school classes with 14-year-old pupils and 3 grammar school classes with 17-year-old pupils). All the selected schools were ordinary schools without any specialization; the classes were characterized as average or even slightly below average by their teachers.

The participating teachers were provided with about 30 problems that can be solved efficiently using one of heuristic strategies:

- Basic school: Analogy, Guess – check – revise and Systematic experimentation;
- Grammar school: Problem reformulation, Solution drawing and Use of graphs of functions.

The pupils sat a written 40-minute pre-test and post-test at the beginning and the end of the experiment (4–5 problems). The problems in both tests were the same. The selected heuristic strategy was the most efficient strategy of solving the problem. Calculators and computers were available on pupils' desks. All the pupils had basic skills in use of spreadsheets in Excel.

Long-term experiment

The long-term, sixteen month experiment was conducted in four classes: Grammar school in Prague (20 pupils, age 16–18), Grammar school in Hořovice (24 pupils, age 12–14), Basic school in Ústí nad Labem (18 pupils, age 14–16), Basic school in Prague (8 pupils, age 14–16).

For the experiment, 200 problems illustrating the use of individual heuristic strategies were created.

Pre- and post-experiment tests consisted of 8 problems (a heuristic strategy was always the most efficient solving strategy). The tests were different for each of the classes; they respected the pupils' age level and knowledge. The problems in the initial and the final tests were identical. The test problems were not presented to the pupils during the experiment, and were not discussed even after the initial test. All the problems from the test were analysed and assessed in detail. Each solution was coded by a member of the research team with respect to the following phenomena:

- way of solving the problem (straight way or heuristic strategy),
- problem-solving mode (arithmetical, algebraic, graphical),
- success rate of problem solving (successfully/unsuccessfully),
- "blank sheet" (the pupil did not even try to solve the task),
- non-evaluable response,
- misunderstanding the question.

Before the experiment started, all the participating pupils were tested and assessed in all four components of CPS. The testing was carried out again post experiment with the exception of Váňa's intelligence test of intelligence as, according to the psychologists, no significant changes in intelligence could be expected.

Cooperation between the teachers and the research team was very intensive and systematic and was going on for the period of two years. Each of the teachers was cooperating closely with one member of the research team. Apart from conducting the experimental teaching, the teachers also collected pupils' worksheets with solutions of the problems and evaluated them. They were continuously observing the pupils and kept record of these observations. The observations focused on changes in approaches to problem solving and pupils' success rate in solving problems in general, not just in experimental problems. Regular meetings of the teachers with the respective researchers were usually held once in two weeks. The following issues were discussed: worksheets, individual problems, strategies used and the individual pupils' responses. The teachers also sent a brief report by email once a week. The members of the research team had access to the pupils' worksheets during the whole experiment. They used them for enriching the existing problems by new procedures that were spontaneously developed in the lessons. Moreover, the worksheets served as feedback with respect to the success rate of the solutions.

Once in six months the cooperating researcher came to one of the lessons from the teaching experiment and once or twice during the whole experiment a video recording of the teaching unit was made.

The experiment was concluded by structured interviews with the participating teachers. Also some reactions of pupils to the use of heuristic strategies in teaching were collected.

3. Results and discussion

3.1. Short-term experiment

When comparing the use of heuristic strategies by pupils in the pre-test and post-test, the following was detected:

Basic schools pupils (supported strategies Analogy, Introduction of an auxiliary element, Guess – check – revise and Systematic experimentation):

- Analogy: The four month period was too short;
- Guess – check – revise and Systematic experimentation: The growth in the use of this strategy was 30%.
- Introduction of an auxiliary element: The growth in the use of this strategy was 50%.

Grammar school pupils (supported strategies Problem reformulation, Solution drawing and Use of graphs of functions):

- Problem reformulation and Solution drawing: The four month period was too short;
- Use of graphs of functions: The growth in the use of this strategy was 50%.

It can be stated that even the short period of time in which heuristic strategies were used by pupils was sufficient to bring positive changes in pupils' attitudes to problem solving on both types of schools. Moreover, almost all problems solved using these solving strategies in the final test were solved correctly.

Teachers' observations can be summarised as follows:

- About one half of pupils stopped being afraid to solve word problems at the end of the experiment, they stopped withdrawing from the solution in case they were not sure of how to solve them from the very beginning.
- They learned to look for the solution, not to give up.
- Also pupils and students who used to be passive in lessons of mathematics started to get involved in problem solving.

3.2. Long-term experiment

Increased *frequency of the used strategies* was detected.

We observed a decreased *frequency of unsolved problems.* It can be concluded that using suitable heuristic strategies played a role in the pupils' decision at least to try the solution.

The following was detected in the use of heuristic strategies:

- Experimental strategies (Systematic experimentation, Guess – check – revise) and Working backwards were the only chosen by the pupils spontaneously also at the beginning of the experiment.
- The most considerable increase in the use of heuristic strategies was in cases of Systematic experimentation, Solution drawing, Use of graphs of functions and Introduction of an auxiliary element.
- The pupils were almost always successful when using the strategies Systematic experimentation and Guess – check – revise.
- Introduction of an auxiliary element: about one half of the pupils were successful in the final test.
- The (albeit sporadic) use of Analogy, Omitting a condition, Specification and generalisation and Problem reformulation in the final tests was successful.

In the course of the experiment, the pupils showed improvement in two of the components of CPS. All the pupils showed some but moderate improvement in the component *Reading comprehension*. The pupils from all the classes considerably improved in the component *Creativity*. A more detailed inquiry shows the highest degree of improvement in the area of fluency and flexibility. In case of Ability to use existing knowledge no statistically significant changes could be observed. The tools used for determining pupils' CPS do not allow us to separate the impact of the teaching experiment and the pupils' natural development completely; however, the psychologists claim the growth in the studied areas was higher than can be ascribed merely to pupils' natural development over the period of 16 months.

The following can be concluded from structured interviews with the teachers:

- Analogy is relatively popular among the pupils in problems that can be reformulated using more "user-friendly" objects, e.g. numbers. It is regarded by teachers as potentially useful for solving other than mathematical problems.
- Working backwards can be learnt by pupils relatively easily. Clever children select it spontaneously as the first way of solving a problem in appropriate situations.

- Specification and generalisation is a strategy useful not only for solving problems in mathematics, it can be also used in other subjects, e.g. physics.
- If pupils are to be able to use the strategies Problem reformulation, Omitting a condition, Generalisation and specification and Decomposition into simpler cases, they have to solve a relatively large number of problems with their teacher; this was not achieved in the experiment. As far as the strategy Introduction of an auxiliary element is concerned, pupils also need a relatively high number of problems to master it actively. In the teaching experiment this was achieved in case of problems from geometry.

Pupils' assessment of heuristic strategies can be summarised as follows:

- Systematic experimentation can be used with a great variety of problems, its use is simple, and a computer can be used with it.
- Guess – check – revise is a fast way to finding the solution if computer is not available.
- Working backwards is in some problems the easiest way to finding the solution.
- When using the strategy Introduction of an auxiliary element in geometry, it is helpful to make an illustrative picture and mark in the picture as much as possible. GeoGebra helps a lot at this stage.
- When using the strategy Analogy, it works well to pose a simpler problem with more "user-friendly" numbers. This helps the solver realize how to solve the original problem.

The experiment also brought some results related to the use of information technology (IT) when solving problems using heuristic strategies. These can be summarised as follows:

- Pupils learned to use IT in the strategy Systematic experimentation very quickly.
- They grew more confident in selecting the initial value in Guess – check – revise sensibly already after 3 months.
- Pupils successfully applied the strategy Systematic experimentation in solving problems whose solution through equations would have been too difficult or impossible.

- Problems where pupils use IT to formulate or discover a hypothesis about a possible solution are very attractive for pupils. These include both problems solved using spreadsheets and problems from geometry solved using dynamic geometry software.

The experiment did not have impact only on the pupils. It also had impact on the participating teachers. They lowered their demands on accuracy and correctness in their pupils' communication and recording in favour of understanding the problem solving procedures, showed more tolerance to variety in pupils' solutions, acknowledged a change in their attitude to mathematics teaching to using constructivist and inquiry-based approaches, and started to pose their own problems with the aim of making the pupils understand the various strategies better.

4. Concluding remarks

The prediction that at post-experiment pupils would be able to use some heuristic strategies in problem solving actively was confirmed. This is true for the following strategies: Systematic experimentation, Introduction of an auxiliary element, Solution drawing, Use of graphs of functions, Guess – check – revise and Working backwards.

During the experiment, the pupils improved in two components of CPS. All the pupils showed some but moderate improvement in the component *Reading comprehension*. The pupils from all the classes considerably improved in the component *Creativity*. A more detailed inquiry shows the highest degree of improvement in the area of fluency and flexibility.

The following can be considered one of the most important results of the project: The pupils ceased to fear problem solving and did not put them off if they could not see the solving procedure immediately. They stopped withdrawing from the solution in case they were not sure of how to solve them from the very beginning. They learned to look for the solution, not to give up. This could be observed in about one half of the participating pupils.

As far as the impact of the experiment on the participating teachers is concerned, it can be claimed that they show more tolerance to variety in pupils' solutions. They are more interested in their thinking processes while solving problems. They admitted that their attitude to teaching mathematics has changed to more constructivist, inquiry-based approach.

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Comprehension of elements of combinatorics in real-life situations among primary school students

Abstract. Mathematical competence has been identified at the EU level as a key competence to be developed in primary school. Elements of combinatorics and the probability theory are a trending topic in the current pedagogical discourse on teaching mathematics in Latvian primary schools. The expected learning outcomes for this topic include improved and practically applicable problem-solving abilities among students. An integrated approach to teaching this topic helps students comprehend the need for mathematics in real-life situations. In this article, we analyze mathematics curricula and discuss the problems associated with primary school students' understanding of how to use combinatorial elements in different real-life situations. These considerations are examined in the light of findings from a student and teacher opinion survey as well as national diagnostic test results for students in third, sixth, and ninth grades (2012-2014). Implications are drawn about ways to improve the Latvian mathematics curriculum in grades 1-6 and methodological recommendations are proposed for teaching different mathematical problem-solving strategies, such as guess-and-check, tables, and graphs.

1. Introduction

A modern, knowledge-based economy requires people with appropriate skills and competencies. Mathematical competence is included among the eight key competences in European schools (Eurydice, 2012) to be developed already at the primary level. Mathematical competence has been

Key words and phrases: competence, elements of combinatorics in primary school, real-life situations, problem-solving strategies..

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identified at the EU level as a key competence for personal fulfillment, active citizenship, social inclusion, and employability in the knowledge society of the 21st century (Eurydice, 2011). Moreover, current trends in European education involve integrated lifelong learning strategies (EU Council conclusions, 2010) and education for sustainable development (Salite, 2008).

Elements of combinatorics (hereinafter – EC) and the probability theory are a trending topic in the current pedagogical discourse on teaching mathematics in Latvian primary schools. At this stage of learning, mathematics should be tied to practical real-life issues (Sawyer, 2008). Tasks with EC are particularly rich in practical examples (Anderson et al., 2004; Roberts et al., 2005; Płocki, 2004). Though they may be unaware of it, primary school students generally have multiple experiences with real-life situations which feature EC. For instance, children use door codes, play dice games, make arrangements and combinations, perform sampling procedures (ordered sampling with/without replacement or non-ordered sampling with/without replacement), and so on. Life is a matter of chance and contingency. The key to predicting the probability of events is knowledge of combinatorics and the probability theory. These considerations underscore the importance of teaching EC and the basics of the probability theory to primary school students with due reference to real-life situations.

This study seeks to address the problem of low academic achievement among primary school students, which was exposed in an international study. Thus, the results of the 5th cycle of the Program for International Student Assessment (PISA 2012) suggest that the lowest achievement in mathematics among Latvian students is demonstrated in topics such as probability and statistics, with scores up to 12 points below the average OECD result (Geske et al. 2013).

This study aims to investigate the possible ways of improving the Latvian mathematics curriculum in grades 1 to 6, with an aim of preparing students for propaedeutic learning of EC. The study explores which aspects of the topic "Elements of combinatorics and the probability theory" should be emphasized in the primary school curriculum of mathematics and subsequently focused on while teaching mathematics to primary school students.

2. Research methodology

In order to determine the changes which are warranted in the primary school curriculum of mathematics in Latvia, an evaluative case study (Geske and Grinfelds, 2006) was chosen. Then, an analysis of relevant documents was performed. These documents include the Latvian National Standard for Primary Education and Mathematics Curriculum Samples for grades 1-9 (MK, 2014). Relevant experience from other countries was appraised with regards to the topic "EC and the probability theory" in mathematics curricula and textbooks. Since the National Centre for Education (NCE) of the Republic of Latvia administrates the state examination system in Latvia, from the drawing up of tasks to the descriptive analysis of national tests and exam results, we used the statistics summarized by NCE of the results of the national diagnostic tests in mathematics for grades 3 and 6 as well as exams for grade 9 in years 2012-2014 regarding the students' ability to solve tasks with EC and the probability theory (VISC, 2012a, 2013a, 2014a). For the purpose of the pilot study, the authors prepared questionnaires and organized a teachers' opinion survey as well as a University students' survey. 130 primary school mathematics teachers from different regions of Latvia took part in the teacher survey in March, June and August, 2015, which was conducted at the teachers' further education courses. The teachers' opinion survey was used to appraise their views on the primary school students' interest in and experience with EC in different real-life situations. Also, the teachers' opinions were canvassed regarding the teaching of EC and helping students master age-appropriate problem-solving strategies for tasks with EC. Daugavpils University (DU) students took part in a students' survey in February, 2015. They were students from randomly selected groups with different specialties, from two faculties Faculty of Social Sciences, and Faculty of Natural Sciences and Mathematics) . The number of completed questionnaires was 74. A small random sample of University students (from only one University) was surveyed, therefore the results are only used to reveal the general tendencies. The DU student survey was used to evaluate the constancy of knowledge and skills related to EC and the probability theory a few years after the completion of secondary education.

The research methods include:

- Comparative analysis of mathematics curricula in different countries (Latvia, Poland, Russian Federation, and USA) with a special focus on the topic of EC and relevant skills to be acquired in grades 1-9 (with children aged 7-15);
- Analysis of Latvian student scores in national mathematics tests (2012-2014) in grades 3, 6, and 9 with a special focus on the students' skills of solving tasks with EC;
- Analysis of findings from the student and teacher opinion surveys.

3. Theoretical background

The teaching of combinatorics is closely associated with the probability theory and statistics: combinatorics is a branch of mathematics concerning the study of finite or countable discrete structures, and enumerative combinatorics includes counting the structures of a given kind and size (Anderson et al., 2004; Mencis et al., 1993; Płocki, 1992; Płocki, 2004).

Garfield et al. (1988) summarized recommendations from teachers regarding more efficient teaching and learning of stochastics. They begin with emphasizing that the teachers should introduce topics through relevant activities and simulations rather than abstractions. Also, the teachers should convince their students that mathematics is a useful practical tool in real-life situations rather than an abstract system of symbols and rules. Batanero et al. (1997) discuss two essential components in the teaching and assessment of combinatorics (basic combinatorial concepts and models) and five combinatorial procedures:

- Logical procedures: classification, systematic enumeration, inclusion/exclusion principle, recurrence;
- Graphical procedures: tree diagrams, graphs;
- Numerical procedures: addition, multiplication and division principles, combinatorial and factorial numbers, Pascal's triangle, difference equations;
- Tabular procedures: constructing a table, arrays;
- Algebraic procedures: generating functions.

According to Batanero et al. (1997), manipulative materials and tree diagrams as well as meaningful activities linked to the probability theory can and should be used with children in primary school, including very young students. Combinatorial reasoning is not restricted to solving verbal combination and arrangement problems; it includes a wide range of concepts and problem-solving abilities (Batanero et al., 1997).

According to the pedagogical and psychological considerations in the relevant EU documents of the educational policy (Cedefop, 2008; Eurydice 2011, 2012), the students need to be aware of the skills and competences they develop. The modern teaching and learning processes accommodate different approaches. The competence approach focuses on the outcomes (Grootings and Nielsen, 2008). Learning new knowledge and skills is best achieved with the constructivist approach (Bruner, 1977), which requires involving the students in the discovery of new knowledge. The integrated approach (Thomas, 2000) is used when the teacher links didactic tasks with real-life situations. The creative solution of problems that feature EC requires critical thinking and divergent thinking (Dewey, 1993; Collins and Amabile, 1999; Kolb, 1984; Savery, 2006). Choosing the appropriate approaches, methods, and strategies is an essential aspect of a teacher's pedagogical competence.

4. Research results

4.1. EC in Latvian primary education standard and curricula

Primary school students are expected to gain problem-solving experience and learn to make sound mathematical judgments. The Latvian National Standard for Primary Education and Mathematics Curriculum Samples for grades 1-9 include such topics as "elements of information processing, statistics, and the probability theory" whereby students learn "to gather, process and analyze information; group related elements and make sense of the concept of probability" (MK, 2014). Mathematics curriculum samples for grades 1-3 suggest that the students learn such skills as making comparisons, sorting, arranging objects according to given or independently identified features, and reading tables, bar diagrams, and texts. In grades 4-6, the focus shifts to elements of statistics, as students learn to collect and record research data from surveys, sort and systematize spokendata, as well as create visual representations of their findings. Meanwhile, the mathematics curriculum for grades 7-9 includes the topic "Discrete models," which encompasses EC and the probability theory:

- Probability, set, possible outcomes, favorable outcomes; computing probabilities of events when sample space is finite;
- Samples with certain properties, their creation; identifying combinatorial properties of objects and/or the number of objects by means of logical procedures; applying the rule of sum or addition principle and the rule of product or multiplication principle to count the number of different samples.

In Latvian mathematics textbooks for grades 1-3, the presence of EC is sketchy. Textbooks for grades 4-6 also feature only random tasks with EC. Such tasks are included in the teaching aids for mathematical contests and competitions. They are used during extracurricular classes with students who show a special interest in mathematics. The Latvian mathematics curriculum could benefit from foreign experience.

4.2. EC in the primary school mathematics curricula of different countries

To appraise the experience of teaching and learning EC in other countries, we examined some primary school mathematics curricula from the US, Poland, and Russia. More specifically, we focused on the topic "EC, the probability theory, and statistics" in terms of its contents and relevant skills in grades 1 to 9 (with children aged 7 to 15). The US curriculum (NCTM, 2000) introduces the topic gradually, from primary up to secondary school. Even preschoolers are taught to estimate events in terms of them being more or less probable (without actually calculating the probability). In grades consistent to Latvian grades 3 to 5, the students are taught that the probability of events is expressed in numbers. The Russian mathematics curriculum for grades equivalent to Latvian grades 5 to 9 (Gimadieva, 2012) features a comprehensive, detailed list of concepts, from EC to calculating the probability of events with classic statistical and geometric formulae. Moreover, some schools in the Russian Federation offer an extensive course of combinatorics and the probability theory already in grade 2, such as "Learning to solve combinatorial problems" (Programma, 2013). Mathematics curricula in Poland (Podstawa Programowa, 2013) for grades equivalent to grades 7 to 9 give an introduction to the probability theory.

Thus, mathematics curricula in different countries are similar in as much as the statistical elements, combinatorics, and the probability theory feature in all curricula. In all four countries, the greatest focus on combinatorics and the probability theory is for grades consistent with Latvian grades 7-9 of primary school, although elements of statistics, such as the reading and drawing of charts, are already incorporated in grades 1-6. Meanwhile, the differences are related to the breadth of approach to the subject matter, the progressiveness of teaching it, and the time when these topics are first addressed (i.e. how old the students are when they first encounter concepts from EC and the probability theory). A propaedeutic course is somewhat more developed in the USA and in the Russian Federation. The above experience suggests that concepts from EC and the probability theory tend to be introduced with their appropriate usage being taught already in preschool.

4.3. Analysis of results of exercises with probability theory problems from national tests in mathematics (2012-2014) in primary school

To determine how the students succeed in reaching the learning outcomes stipulated in the national primary education standard for mathematics, we analyzed their performance in national tests. The standard mathematical competence of Latvian students in the National Standard for Primary Education is stipulated in three stages of learning: grades 1-3, grades 4-6, grades 7-9 (MK, 2014). In this study, we analyze the national tests in mathematics, which feature tasks with EC. Each year, the National Centre for Education performs a descriptive analysis of all state exams or test results. We relied on their digests of student achievement in national mathematics tests for grades 3, 6, and 9 in 2012, 2013, and 2014 (VISC, 2012a; 2013a, 2014a).

In 2014, third-graders had to solve a combinatorial task for the first time since 2012, see Exercise 9 (VISC, 2014c).

Exercise 9. There are 20 **different** mushrooms in a basket. The number of king boletes **is the smallest**. The number of russulas is **greater by** 4 than that of chanterelles. How many mushrooms of **each kind can there be** in the basket?

Solving this combinatorial task required the comprehension of several concepts (highlighted in bold). In addition, the third-graders had to write down their answers in a table, and the task had two possible solutions. The third-graders' academic achievement in exercise 9 (VISC, 2014a) was the following: 30.64% of students found both answers, whereas 41.52% gave only one of the two possible answers. The total success rate in the diagnostic test for grade 3 was 77.54%, total number of students was $N = 16767$ (VISC, 2014a). It follows that the students do well in traditional tasks but struggle with non-standard ones which feature EC. Apparently, they lack the experience of putting the solution in a table or providing it descriptively. According to the findings from the teacher survey, not all students are able to use the guess-and-check strategy, which would have been useful in exercise 9.

In 2012-2014, mathematic tests for grade 6 contained tasks with elements of statistics, but they will not be analyzed here.

At the end of grade 9, students have a two-part national exam in mathematics. The first part consists of the test part with simple, brief tasks. The second features standard tasks that resemble those found in textbooks, but the last task in the second part requires the application of knowledge and skills in new, non-standard situations. In 2012, the first part of the exam for grade 9 featured a simple exercise to calculate the probability of an event, but there were no average performance indicators for this exercise (VISC, 2012a, 2012b). In 2013, the second part of the exam for grade 9 had the following task (VISC, 2013b):

Exercise 4. Andrew wants to buy some ice-cream. The supermarket sells wafer cones and a selection of banana, strawberry chocolate, caramel, and vanilla flavors. The customer can ask for two balls of any flavor (identical or different).

a) List all possible combinations for a wafer cone with two ice-cream balls.

b) Calculate how many different wafer cones with two ice-cream balls can be bought in this store. The order of balls in the cones is irrelevant.

c) What is the probability of Andrew buying a wafer cone with one strawberry-flavored and one banana-flavored ice-cream ball, if the balls are randomly selected?

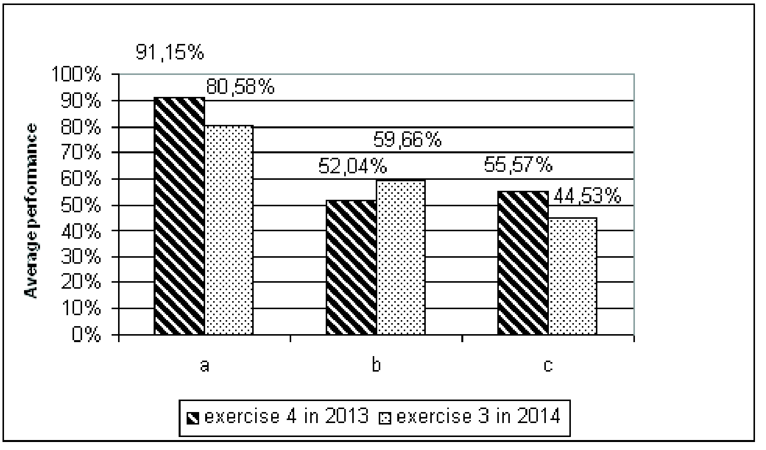

Figure 1. Academic achievement for exercise 4 (N = 15890) in 2013 and exercise 3 (N = 14797) in 2014 from mathematics exam in grade 9.

In 2014, exercise 3 was similar to exercise 4 from 2013, but this time variant 3*c* was much more complicated than 3*b* (VISC, 2013b, 2014b). The ninth-graders' success rates in exercises 4*a*, 4*b*, and 4*c* (2013) and exercises 3*a*, 3*b*, and 3*c* (2014) are presented in Figure 1 (VISC, 2013a, 2014a), and the total number of students was: 15 890 in 2013 and 14 797 in 2014. In 2013, 91% of students solved the simpler task (4*a*), while only approximately 50% managed to solve the more complicated ones (4*b* and 4*c*). The trend continued in 2014: 80% of students solved the simpler task (3a) with success rates dropping to 40-60% in more complex tasks (3*b* and 3*c*). Some ninth-graders failed to grasp the method of identifying combinations with replacement (in 2013) and without replacement (in 2014), and did not seem to comprehend the concept of probability. Some students made calculation errors too.

4.4. Results of student survey

A test featuring seven multiple choice questions (tasks) on EC and the probability theory and one question – to indicate familiar strategies for solving combinatorial problems – was administered to DU students, who had studied EC and the introduction to the probability theory in secondary school. Their ability to solve tasks with EC was tested a few years after graduating from school with the aim of determining the stability of their knowledge and current comprehension of EC in real-life situations.

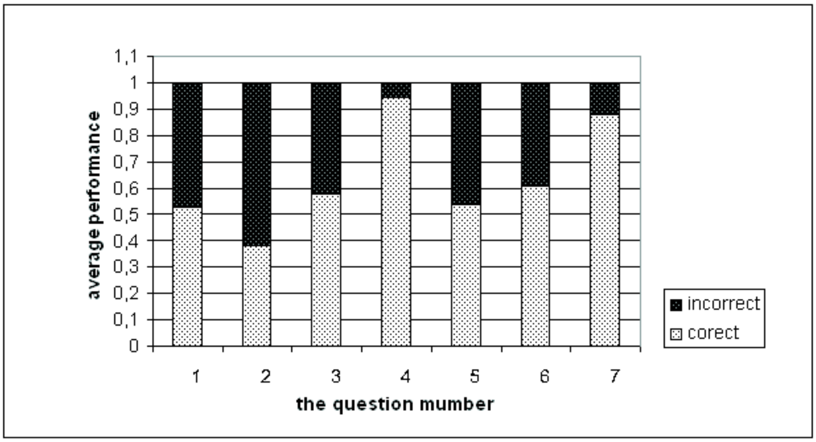

Figure 2. DU students' average performance in a test with seven questions ($N = 74$).

The DU students' average performance in this test containing seven questions is illustrated in Figure 2. Question 2 turned out to be the most challenging – the students were asked to state the number of the possible three-digit combinations for a door code, with none of the digits repeating within a single combination (i.e., all buttons need to be pressed at the same time, so changing the order of digits does not generate a different code). Prevalent errors appear to suggest that many respondents seem to have failed to grasp the concept – at the same time. In question 6, the respondents had to identify an impossible event, and most chose the answer "In Latvia, a tulip is likely to bloom in January" rather than the option "Selecting two arbitrary digits, their sum is 20". The respondents may have confused the concept of a digit and the concept of a number. On a more positive note, an overwhelming majority (85%) gave the correct answers to question 7, where they had to state the likelihood of winning with a randomly selected instant lottery ticket. Also, most respondents gave the correct answers to question 4, where the real-life situation involved a combinatorics problem which could be solved by applying a multiplication principle.

Finally, question 8 of the students' survey had four multiple choice options, and the respondents could have chosen more than one familiar task-solution strategy. Out of the four specific strategies offered for solving combinatorial tasks, the students appear to be the least familiar with the graph construction strategy.

A random sample of Daugavpils University students from two faculties indicates that the students' ability to solve simple tasks with EC a few years after the completion of secondary education is sufficient, although some university students struggled with reading comprehension and understanding mathematical concepts. Comparing national mathematics test results (2012-2014) in primary school against findings from the DU students' survey confirms that both primary school students and university students struggle with tasks that feature combinatorial problems. It means that the teachers should focus on helping the learners master appropriate task-solution strategies and comprehend relevant concepts.

4.5. Results of teachers' opinion survey

The teachers were asked to evaluate a list of 10 problems related to the application of EC in real life. The most topical real-life situations for primary school students in Latvia according to the teachers' point of view are: *dealing with Internet security* (36%)*; secrets of text encryption* (35%)*; in how many ways can a group of students (who will be appointed to the school board) be selected from the class* (19%). The less popular questions that the students discussed with the teachers are: *How many options can there be for door codes? Why, since 2006/2007, have 8-digit phone numbers been introduced in Latvia? What makes a password secure for a bank operation, or computers?* The teachers' opinion survey reveals that primary school students are interested in problems related to the application of EC in real life.

We have adapted the above-mentioned combinatorial procedures for lower primary school as well as survey questions to elicit the teachers' understanding of these specific strategies for solving tasks with EC. Findings from the teachers' opinion survey suggest that from the four suggested specific strategies for solving a combinatorial task (logical procedures, constructing a table, tree diagrams, and graphs), they know the least about the graphs strategy. Although logical procedures did not have the lowest score (45% claim to know them), only 39% of primary school teachers admit to have introduced the guess-and-check strategy to their students. As mentioned above, poor knowledge of the guess-and-check strategy and the inability to use tables may account for the fact that only a fraction of the third-graders managed to answer question 9 from the diagnostic test in 2014 (VISC, 2014a, 2014c). Reading comprehension and planning the course of solving word problems were most often cited as the reasons for poor student performance when solving mathematical word problems (85% and 60% of teachers, respectively; the survey question *What poses difficulties to students in solving mathematical word problems*?)

A group of 77% of primary school teachers support the idea of drafting a propaedeutic course on combinatorial concepts and problem-solving strategies for students in grades 1 to 6. The teachers' answers suggest that they need methodological guidance on teaching EC in grades 1 to 6. The teachers need assistance with drawing parallels with real-life situations when explaining concepts such as certain or impossible events, more likely or less likely. Another concern is insufficient diversity in question formulations: *how many different options (variants, cases)…, how many choices…, what is the best bargain…?* We suggest that the teachers try to create tasks about real-life situations which require the application of different task-solving strategies and modeling of problems. Practical tasks in grades 1 to 3 already require the usage of the addition principle, but the students can also learn that tasks may be solved with the help of tree diagrams, tables, and the guess-and-check strategy.

5. Discussion and conclusion

Our lives are guided by chance as much as they follow certain laws of logic. Therefore, humans crave understanding of the probability of events. They want to know the odds when making choices. Teaching and learning mathematics in primary school should be relevant to the realities of daily life. Such connections would create and reinforce interest in learning mathematics. EC is a mathematical theory that helps address the practical issues of daily life and, thus, can help create and sustain the students' interest in mathematics.

Solving problem tasks during regular classwork should involve the analysis of non-standard situations which require varied solutions. Tasks with EC for grades 1 to 6 can be considered non-standard, since they invite students to examine different cases, use their imagination to visualize the situation, and make original judgments.

The research results indicate that students have difficulties with solving combinatorial problems related to real-life situations. This is confirmed by the analysis of the teachers' opinion survey and the analysis of Latvian student scores in national mathematics tests (2012-2014) in grades 3 and 9, especially the students' skills regarding solving problems with EC, which call for an initiative to improve the mathematics curricula and the methodology for teaching EC in lower primary school. The analysis of relevant experience from a number of countries exposes the need for a propaedeutic course in Latvian primary education. This course would gradually and systematically introduce EC to primary school students in grades 1 to 6 and prepare them for the in-depth learning of this topic in grades 7 to 9 and later on in secondary school.

In view of the fact that the students' skills of text analysis are insufficient, mathematics curriculum for grades 1 to 3 and 4 to 6 (more specifically, the sections "Mathematical analysis of natural and social processes" and "Creating and studying mathematical models with mathematical methods") need to be adjusted by adding concepts from EC and teaching the students their proper usage in different school subjects. For instance:

- often-rarely, always-never, sometimes, occasionally, at the same time,
- event, likely-unlikely; certain, impossible event; more or less probable event,
- mutually exclusive events,
- choice, sample, ordered-random, different types and
- chance-regularity.

Tasks which require the application of EC in real-life situations should draw the students' attention to different task-solving strategies, such as:

- Logical procedures: guess-and-check, predict-check-prove, modeling with counting material.
- Graphical procedures: tree diagrams, graphs, etc.
- Tabular procedures: constructing a table.

The pedagogical modeling of real-life situations should involve didactic games, board games, recreational games, experiments, etc., thereby encouraging the students to perceive and discover regularities, generalize, and critically evaluate different options.

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Development of students' beliefs in mathematical understanding in relationship to mathematics and its application

Abstract. The study investigates the development of relationships between epistemological beliefs and the perception of mathematics in the course of a year. The intervention consisted of three supplemental courses with 16 students of grade 11 and 12 in German high schools, two devoted to coding theory and cryptography, and one to the mathematical aspects of cosmology and particle physics.

During the first quarter of the school year, the students studied the mathematical foundations of the course topics. In each of the last three quarters, the students were offered a choice between several project topics, or could find a topic themselves. They worked on the topics alone or in pairs. At the end of each quarter, the students had to present their results.

At the end of the second and fourth quarter, the students were interviewed, following a semi-structured concept. The first interviews showed that students' opinions about mathematical understanding were related to their definition of mathematics and their choice of project topics. By dividing mathematical understanding into active and passive categories, it became obvious that the students who used abstract attributes defined mathematics as suitable for applications and chose an abstract topic for projects, and vice versa. As it became evident in the later interviews, most students' understanding of mathematics changed when their definition of mathematics and their choice of topics were completed, both from an application and an abstract point of view during the third and fourth quarter of the year.

Key words and phrases: mathematical understanding, abstract mathematics, application, beliefs.

AMS (2000) Subject Classification: Primary 97C99, Secondary 97D20.

1. Introduction

Teachers often think about their students' understanding and motivation and how to increase them. Students are aware that they are expected to understand mathematics, often would like to do so, and try to fulfil expectations. The achievement of goals and expectations will affect their perception of mathematics, their beliefs and their mathematical competences in the future. With help of our results, teachers might be able to influence students' relationship to mathematics and the possibility to rectify their motivation for mathematics.

The study investigates the development of students' epistemological beliefs (Liu and Liu, 2011; Schommer-Aikins, et al., 2000; Maaß, 2004), perception of mathematics and understanding, the choice of project topics, the editing of projects and relationships between the aspects over a year at school. Changes in the points mentioned above are measured in relationship with respect to different forms of work and types of topics for lessons and projects.

The article is based upon a study of 16 students of grade 11 in German grammar schools taught by the author over a whole school year in 2013/2014. In the middle and at the end of the school year semi-structured interviews were conducted.

The aspects of the investigation mentioned above lead to the following research questions:

1. How can we categorize students' opinion about mathematical understanding, their definition of mathematics, their choice and work on topics and the relationships between them?
2. How can we describe the development of these during the school year?
3. Can the beliefs in mathematics be influenced by relationships between the aspects in rq 1?

As mentioned by Smith (2014), there already exist a lot of investigations of beliefs of students in mathematics of the qualities and skills and also whether mathematics is useful. Kloosterman, Raymond and Emenaker (1996) describe an investigation concerning students' beliefs of grade 1 to 4 over three years. Furthermore results concerning students' beliefs in learning mathematics are given by Kloosterman and Cougan (1994), Tahir and Adu Baker (2009) and others. A study concerning relationships between epistemological beliefs is given by Stockton (2010). Up to now no study concerning the questions 1, 2 and 3 over a whole year at school exists.

2. Structure of courses

According to the Conference of the Ministers of Education of the Länder in the Federal Republic of Germany (Kultusministerkonferenz, 2013), students can improve their ability to study with help of a "special learning performance" (*besondere Lernleistung*). As response some of the federal states built special courses in the last three years before graduation. In North Rhine-Westphalia (NRW) those project courses started in the school year 2011/2012 and take place for a year during the last two years before Abitur (Ministry of Education in NRW, 2010). By the Ministry of Education in NRW, students should have the possibility to study autonomously and cooperatively in connection with projects and applications as well as in interdisciplinary contexts.

Because of shortage in mathematics teachers[1], a lot of schools were unable to establish project courses. Inspired hereby the Institute of Mathematics and Computer Science at the University of Münster in connection with the Institute of Education of Mathematics and Computer Science established project courses of different topics in connection with the study described above.

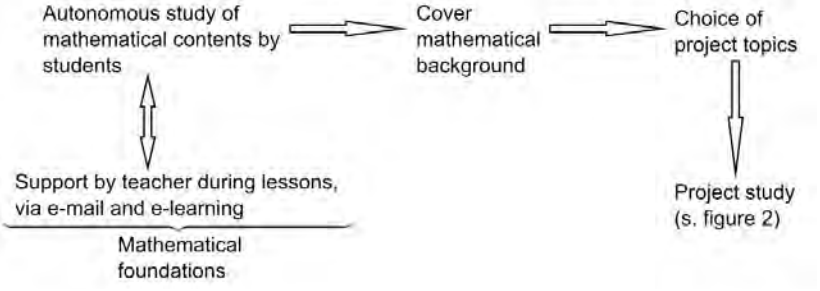

Figure 1. Phases of the course in the first quarter of the school year.

The courses were taught by the author every two weeks for two up to three hours. During the first three months of the beginning of the school year, the courses worked on the mathematical foundations of the course topics: *coding and cryptography* or *particle physics and cosmology*[2]. For example the mathematical foundations of coding and cryptography are numbers, groups, rings and fields, matrices, basics of topology, plane al-

1. These shortages appear besides mathematics in all sciences and computer science.
2. From now on we will reduce the description to *coding and cryptology* to make it clearly.

gebraic curves, probability and different algorithms. During this phase foundations were set during the lessons, and the students had to study autonomously mathematical and application-oriented backgrounds in between the lessons, see the left side of figure 1. After mathematical background had been covered, the students were encouraged to look for topics for projects about coding. In case they were unable to find topics themselves, the teacher suggested topics. Except one student all of them were unable to find topics themselves and accepted suggestion of the teacher. After the choice of topics the students started to elaborate their projects for two months and to prepare presentations. Except one group of two members, all of the students worked alone. Later they collected their results alone or two by two.

After choosing their topics, the students began to process their projects in a way which might be described by cycles. In the beginning they studied the topics autonomous. During the time of four up to six weeks they had to study their topics. All over the time the students were offered support by the teacher via e-mail or by e-learning and were given hints of literature as well as copies. In the end they were expected to prepare a presentation. Afterwards they presented the results of their work up to then to the other students, followed a short discussion including improvement suggestions by the other students and the teacher. The students extended their projects afterwards, see figure 2.

Five months after the beginning of the school year the students presented the final results of their projects for about 15 minutes each. Because the mathematical foundations in the beginning of the school year included the principles of all topics, the students were able to understand the mathematical contents of the presentations.

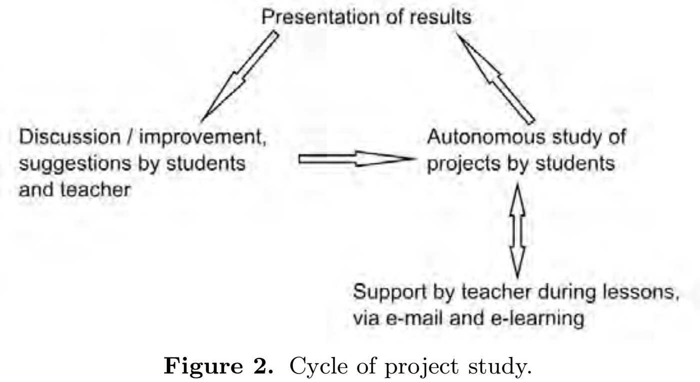

Figure 2. Cycle of project study.

The next phase of lessons began with an introduction to cryptography by lectures of the teacher. Afterwards the students worked on two projects analogous to phases described above and shown in figures 1 and 2. The topics of projects 1, 2 and 3 (s. figure 3) might be connected, e.g. by project 2 *cryptography with elliptic curves* and project 3 *public key in relationship with elliptic curves.* The preparation of each project took two months. The school year finished immediately with the presentation of the last projects.

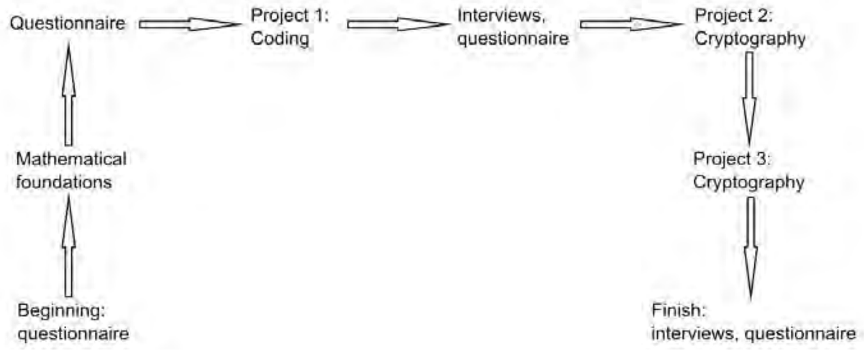

Figure 3. Data collection during the study.

3. Data collection

The data collection is shown in figure 1. Over the school year all students completed four questionnaires and gave two interviews. The questionnaires are created based on Kloosterman and Stage (1992) and Maaß (2004) for aspects of mathematical beliefs and on Schommer-Aikins et al. (2000) concerning epistemological beliefs. All over the year we used the same questionnaires four times. The second questionnaires were filled in at the end of the phase of mathematical foundations for students' projects. The students started the projects about coding immediately after the interviews.

After the presentation of the result of the first projects the students filled in the next questionnaires, and immediate afterwards semi-structured interviews were conducted.

The interviews covered *knowledge and understanding of educational contents*, *perception of mathematical contents*, *association of mathematics and its implementation* and attitudes about statements concerning mathematics: *logical nature, empirical nature, creativity and imagination, discovered vs. invented, socio-cultural aspects, scientific aspects* (Liu and Liu, 2011). The answers are categorized *mathematical understanding, opinion about mathematical background, definition of mathematics, application of mathematics in society* and *meaning of mathematics for nature*. Moreover the interviews contained questions concerning the contents of the courses, where students were asked to give a description of mathematical foundations and their own projects as well as projects by other students.

For this report we concentrate upon the aspects of students' epistemological beliefs in mathematical understanding and students' description of mathematics in relationship to the choice of project topics and the elaboration of projects.

The measurement of mathematical understanding via *reflective thinking* was taken from Stoppel (2012) and Zehavi and Mann (2005) by using the categories *assume, classify, analyse, generalize, concretize, concretize, structure, specialize, form theory, formulate, explain, imagine, remember, imitate, apply* and *execution*.

At the end of the next quarter of the school year the students presented the results of their first projects about cryptography. Afterwards they started new projects about cryptography and presented results. Immediately after the presentation of results students completed the fourth questionnaire and gave the second interviews. The questionnaires were the same as before. The second interviews differed from the first ones in questions about the contents of the course. Students were required to describe contents of their own projects and projects of the other students from all over the school year, not only about the second half.

The interviews were taken by colleagues of the author, the questionnaires were distributed and collected by himself. In order not to influenced by answers of students he did not have a look at the questionnaires and did not listen to the interviews before the end of the school year. Otherwise it might have influenced the assessment.

The first interviews were recorded with 22 students. The second interviews were conducted with only 16, as some of the students did not appear. The interviews of the 16 students are investigated below.

4. Results

During the investigation, attention was attracted by relationships between students' definition of mathematics and their beliefs in mathematical understanding. Also connections between the project topics and students' work appeared (see Stoppel and Neugebauer, 2014). By observing students' definitions of mathematical understanding the definitions are divided into *active* concepts like explanations or applications in contrast to *passive* ones like observations or reproductions. For example an *active* one given by a student is:

> I understand mathematics when I am able to explain something

whereas a *passive* description of mathematics understanding is given by

> Mathematical understanding means, that one is able to reproduce how to achieve [a solution] and why it is this way.

Students defined mathematics in different ways which can be divided into two different types. Some students defined mathematics as an *abstract science*, for example

> Mathematics as support for different (applied) sciences

or

> Mathematics includes axioms, and they are used for conclusions.

Others defined mathematics via *application* as:

> Mathematics is everything all over the world

or

> Foundation of jobs.

The investigation of the students' choice of project topics and their elaboration lead to division into the two categories *abstract mathematical* like

> mathematical backgrounds of the usage of elliptic curves and applications like the usage of RSA [3] focused on applications (not mathematical aspects like the algorithms).

Table 1 shows that 6 of the 16 students defined mathematics via *application* and described mathematical understanding under the usage of *active* reflective thinking, whereas another 7 students defined mathematics as *abstract* and chose *passive* reflective thinking to describe mathematical understanding. Only the remaining 3 students used different types of descriptions. Furthermore all 6 students who defined mathematics in association with application and used active reflective thinking for the description of mathematical understanding chose abstract topics for their coding project. They will be denoted as type 1. Every student who defined mathematics as abstract and described mathematical understanding by passive reflective thinking elaborated applications in his/her project. These students will be denoted as type 2. All cases are presented in table 1.

3. Initials of authors Rivert, Shamir and Adleman.

Definition of mathematics	Reflective thinking	Project topics	Frequency
application	active	abstract	6 (type 1)
abstract	passive	application	7 (type 2)
application	passive	application	1
abstract	active	abstract	1
abstract	active	application	1

Table 1. Results of the first interviews.

For example keywords of a student are given by [S1]:
- Passive reflective thinking: think (yourself);
- Abstract definition of mathematics: local thinking;
- Application as type of project: AES-encryption [4].

The distribution is made visible in figure 4, showing that the types 1 and 2 are opposed as they differ in all three categories.

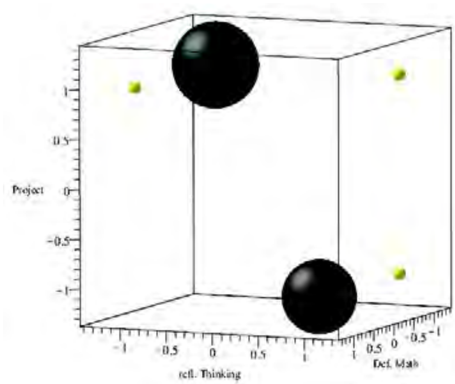

Figure 4. Illustration of frequencies of (definition, reflective thinking, project topics), first interviews.

Changes in relationships between students' definition of mathematics, their opinion about mathematical understanding and the type of project topics appeared in the second interviews. Seven students defined mathematics both in an abstract way and via application. All these 7 students described mathematical understanding by active reflective thinking. Furthermore all of them developed their projects in an abstract way and took a close and exhaustive look at applications. Only one student who defined mathematics in both ways and processed his project topics in an abstract and an application-oriented way described mathematical understanding with passive reflective thinking. All results are visible in table 2 and figure 5.

4. Advanced encryption standard.

Definition of mathematics	Reflective thinking	Project topics	Frequency
application	active	abstract	2
abstract	passive	application	0
application	passive	application	0
abstract	active	abstract	2
abstract	active	application	0
both	passive	both	1
both	active	both	**7**
application	active	application	1
application	passive	abstract	1
abstract	both	abstract	1
abstract	passive	abstract	1

Table 2. Results of the second interviews.

For example student [S1] changed to descriptions:

- Active reflective thinking: explain, reason;
- Abstract and active definition of mathematics: local thinking and algorithm;
- Types of project: application of RSA theory of public key.

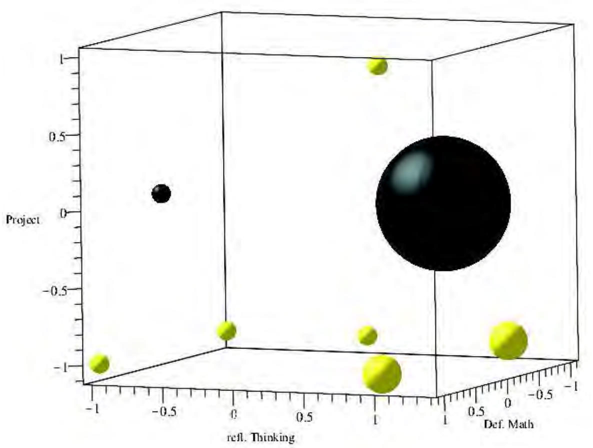

Figure 5. Illustration of frequencies of (definition, reflective thinking, project topics), second interviews.

Answers to the research questions are implied by the results. First the opinions of students about mathematical understanding were related to their definition of mathematics and their choice of project topics. In students' perception of these connections primarily the two different types *active* and *passive* appeared. For their first projects, students who used active attributes defined mathematics as suitable for the usage of applications and chose an abstract topic for projects. On the other hand, the students who used passive attributes defined mathematics in an abstract way and chose applications for projects. During the third and fourth quarter of the year, most students' mathematics understanding changed when their definition of mathematics and their choice of topics were regarded *both* from an application and an abstract point of view.

5. Conclusions

Students' interest in mathematics is influenced by their beliefs of abstract mathematics, its application, mathematical understanding and connections between them. Some opposite types appear between the perception of mathematics and application of mathematics.

A big potential for learning in project work is given by the freedom to elaborate on mathematical projects. The hidden interests of students become visible by connections between their opinion about mathematics, application of mathematics and mathematical understanding.

Based on freedom in processes of projects over a longer period, the opinions of students about mathematics and results of their projects expand to abstract mathematics and application, at the same time when their level of activity was raised. This is reflected by the changes in types of project topics and the results of their autonomous editing.

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Career construction in school mathematics

Edyta Gruszczyk-Kolczyńska

Mathematically gifted children: research results, analysis, conclusions [1]

Abstract. As part of a project, financed by research funding (Project R1700603 *Rozpoznawanie i wspomaganie rozwoju uzdolnień do uczenia się matematyki u starszych przedszkolaków i małych uczniów* [*Assessing and nurturing the development of an aptitude for learning mathematics in older kindergartners and primary school students*]), I have created diagnostic tools for the recognition of the mathematics aptitude of children. I have also assessed the number of mathematically gifted children as well as the highly gifted. I have determined the attributes of their mind and gathered evidence that the aptitude for mathematics declines if it is not nurtured and developed at the right time. I have also taken action in order to help properly accommodate gifted children at school. I will present everything synthetically, as I have to, in the form of the following sections:

✓ The research that prompted the change of heart regarding the existence of an aptitude for mathematics in children.

Key words and phrases: mathematically gifted children; existence of an aptitude for mathematics in children; mental capabilities; mathematics aptitude; older kindergartners; primary school students; evelopment of mathematics ap- titude; children mathematic.

AMS (2000) Subject Classification: 97C30.

As part of a project, financed by research funding , I have created diagnostic tools for the recognition of the mathematics aptitude of children. I have also assessed the number of mathematically gifted children as well as the highly gifted. I have determined the attributes of their mind and gathered evidence that the aptitude for mathematics declines if it is not nurtured and developed at the right time. I have also taken action in order to help properly accommodate gifted children at school. I will present everything synthetically, as I have to. I will include all the information regarding further reading in the footnotes.

- ✓ The mental attributes of mathematically gifted children and the presentation of the results of the study regarding the existence of mathematics aptitude in children.
- ✓ The critical (sensitive) periods of the development of mathematics aptitude in older kindergartners and primary school students. Why aptitude wastes away when not nurtured and developed during critical periods.
- ✓ Why teachers are wrong in their assessment of the mental capabilities of mathematically gifted students.
- ✓ Arguments for the recognition of mathematics aptitude in older kindergartners and a short description of a teacher diagnosis for the recognition of mathematics aptitude in older kindergartners and young primary school students.
- ✓ What has already been done to ensure a better fate of mathematically gifted children and what is the result.

It is commonly assumed that an aptitude for mathematics is visible specifically in older students, when they find a practical use of their advanced knowledge of mathematics. If they manage to attain high scores in mathematics contests, it is obvious that they are highly gifted. It is also assumed that a mathematics aptitude is a rarity, which is why no one is surprised when e.g. only two or three students per form seem to have it. In consequence, it is incorrectly assumed that failures in learning mathematics are caused by the lack of a mathematics aptitude[2].

It is also assumed that children cannot present their mathematics aptitude, as they don't know enough maths. When older kindergartners and generally primary school students seem to acquire maths skills sur-

1. I am making use of fragments of the work *O dzieciach matematycznie uzdolnionych. Książka dla rodziców i nauczycieli* [*Mathematically gifted children. A book for children and parents*] (ed. E. Gruszczyk-Kolczyńska, Wydawnictwo Nowa Era, Warszawa 2012, chapters from first and second parts), as well as the article *Dzieci uzdolnione matematycznie – mity i realia* [*Mathematically gifted children – myths and reality*], parts 1 and 2, "Matematyka. Czasopismo dla nauczycieli" ["Mathematics. A journal for teachers"] 2011, issues 8 and 9.

2. This is a comfortable assumption for everyone. Teachers do not even try to teach better, because"everyone in the class is an artistic mind". The students feel excused for not trying harder, because they think aptitude is intrinsic and hereditary. The parents do not feel guilty for not making sure their children do their homework. They say that the child "took after his grandfather, who was also inept at maths", etc. I have described the actual reasons for school failures in the book *Dzieci ze specyficznymi trudnościami w nauce matematyki. Przyczyny, diagnoza, zajęcia korekcyjno-wyrównawcze* [*Children with specific issues in learning mathematics. Reasons, diagnosis, remediation activities*], WSiP, Warsaw 1992 incl. next issues, part one.

prisingly quickly, the parents and teachers correlate this to their general aptitude and intelligence. This is probably why recognising the mathematics aptitude of children was not pursued.

About the research that prompted the change of heart regarding the existence of an aptitude for mathematics in children

In the 1960s, A. W. Krutetsky [3] – a respected figure in the world of mathematics aptitude – said that the makings of a maths aptitude can be perceived in children. According to Krutetsky, if they are properly fostered, they attain the form described by him in the model of mathematics aptitude in older students. Though it is unclear whether he meant kindergartners or young primary school students.

Almost 30 years later, while assessing the educational effectiveness of the *Dziecięca matematyka* [*Children's mathematics*] program, I have observed that more than half of the children taking part in the program perform noticeably well in mathematics education at school [4]. These children solved maths problems with pleasure, using astonishing maths skills. They also discerned maths problems while e.g. doing housework, taking a walk, doing shopping. Their thoughts concerned numbers and measurement. They wanted to measure, calculate, determine proportions, etc. Such an orientation of the mind was determined by Krutetsky to be one of the more important indicators of an aptitude in mathematics.

This is why, while presenting the results of this study, I have formulated a thesis statement, which says that more than half of the overall number of children manifest their mathematics aptitude, as long as

3. A.W. Krutetsky *Psikhologiia matematicheskikh sposobnostei shkol'nikov* [*Psychology of mathematical abilities of students*], Prosveshchenie, Moscow 1968. In the 80s, I was fortunate enough to be able to attend a few of Krutetsky's lectures, I was also able to have a conversation with him, during which he gave me the book used here. An in-depth perusal of the book has induced me to research the mathematics aptitude of children.

4. This program was carried out in selected kindergartens. When the children from the program enrolled in primary school and finished third grade, I analysed their school performance. The teachers and parents of each of the kids answered these questions: a) *How is the child doing at school? What are their achievements and failures?* b) *Is solving maths tasks visibly enjoyable for the child, and is it easier for him to do so than for his peers?* c) *In real-life situations, does the child express interest in doing calculations and measurements without adult encouragement?* The outcome was that circa 92% of the children in question were successful at school. The teachers were clear that those were their best learners to date. The remaining children (about 8%) had trouble with reading and writing (though they did better at maths). From what was gathered from the parents, about 58% of the kids had a mathematically-focused mind, which is an important indicator of a mathematics aptitude.

proper conditions are maintained at kindergarten and at school. This statement was not taken seriously, as it contradicts the idea that only older children can manifest their mathematics aptitude and that it is a very rare occurrence. In order to prove my statement, I have once more began studying mathematically gifted children a few years later. In 2010, I have finished carrying out a project called *Rozpoznawanie i wspomaganie rozwoju uzdolnień do uczenia się matematyki u starszych przedszkolaków i małych uczniów* [*Recognising and encouraging the development of aptitude for learning mathematics in older kindergartners and primary school students*] [5]. I will begin the presentation of the results of the study by describing the mental attributes of mathematically gifted children.

The mental attributes of mathematically gifted children

By analyzing all math-related activities of children at home, at kindergarten, and at school, I have determined that many of the children exhibit the attributes of mind listed by W. A. Krutetsky in the model of mathematics aptitude [6] in older learners. By my findings, mathematically gifted children stand out among their peers due to:

- ✓ The ease of learning maths skills and understanding anything regarding calculating and measuring [7];
- ✓ Achieving the concrete operation stage (in the J. Piaget sense) earlier than others, exhibiting more precise reasoning [8];

5. A substantial report regarding this study entitled *Wiadomości i umiejętności oraz zarysowujące się uzdolnienia matematyczne starszych przedszkolaków i małych uczniów. Podręcznik, narzędzia diagnostyczne oraz wskazówki do wspomagania rozwoju umysłowego i edukacji uzdolnionych dzieci* [*Information about, and the abilities, and the outlines of the mathematics aptitude of older kindergartners and primary school students. Textbook, diagnostic tools, and tips for the encouragment of mental development and the education of gifted children*] (ed. E. Gruszczyk-Kolczyńska) is located in the Academy of Special Education in Warsaw.

6. Full description of the attributes available in W. A. Krutetsky's aforementioned *Psikhologiia matematicheskikh sposobnostei shkol'nikov*, pp. 201-245. He distinguishes mental attributes which are the basis of the development of any kind of aptitude, as well as the mental attributes of mathematically gifted older students. The description of the former includes the process of solving maths tasks, as this is how the attributes manifest.

7. For instance, to assess that the result of an equation containing solely addition does not rely on the order of operations, the children only need to solve only a few tasks, whereas the non-gifted children will understand the concept after solving several tasks.

8. Gifted four- and five-year-olds make use of operational thinking on the concrete level in a scope just wide enough to develop number concepts and calculating skills. What is more, most children arrive at this level of competence two years later.

- ✓ Easily making sense of situations which require calculating, ordering, determining dependencies, etc. Due to this, they easily perform all the tasks necessary to achieve their goal, spotting errors and properly reacting to absurdities [9];
- ✓ Focusing their attention on sentences for longer periods of time without showing fatigue. Although they will stop upon noticing any sign of ignoring their efforts [10];
- ✓ Doing more attempts at solving tasks if previous attempts were deemed ineffective [11];
- ✓ A creative mindset regarding maths activity. The children look for situations in which they need to make use of calculating, measuring, and organising their environment on their own.

This is why these children know more and have more maths skills than it would seem, given their age and their course of education. It also seems as if they see their environment with maths in mind and seek to mathematise all that surrounds them – they constantly feel the need to count, measure, compare sizes, determine proportions, etc.

Based on these findings, it seems Krutetsky was too cautious claiming that there are only basic components of the mathematics aptitude of children. It is possible that he did not have many opportunities to observe and analyse the functioning of older kindergartners and primary school students in situations where calculating, measurements, etc., were essential to successfully complete a task.

About the creation of tools used for the recognition of the mathematics aptitude of children [12]

In order to begin research on the existence of the mathematics aptitude of children, diagnostic tools needed to be created, as none of the

9. This can be observed in situations where the children are to solve tasks which were poorly phrased on purpose, or are observing adults who intentionally make mistakes while solving maths tasks.

10. While verifying the diagnostic value of the tests used for the recognition of mathematics aptitude, hundreds of tasks had to be solved with the children. Gifted children would do tasks for an hour or more, without showing any signs of boredom or fatigue. But as soon as they noticed any sign of impatience, lack of attention, etc., they refused to continue.

11. If the chosen way of solving a task does lead to any results, they change their way of thinking and try to do it differently, which is often far from the way tasks are solved at school.

12. In-depth information is available in the aforementioned book called *O dzieciach matematycznie uzdolnionych...*, as well as the cited report from the research *Wiadomości i umiejętności oraz zarysowujące się uzdolnienia matematyczne starszych przedszkolaków i małych uczniów...*

existing tools provided for the mental attributes described earlier. I began by analysing what the children were learning at home, at kindergarten, and at school. Based on this, I have delineated 13 areas, which were: spatial awareness, classifications, counting, addition and subtraction, the value of money (buying and selling), measuring length, fluids, mass, and time, geometric intuition, equalities and inequalities, tasks with windows, and intentionally ill-constructed tasks.

For every area, series of diagnostic tasks were constructed, ranging from very easy to hard. Every series of diagnostic tasks was adjusted until the employed statistical procedure showed that the ways of solving the tasks differentiated well enough. That is, older children exhibited higher competence levels than younger children (which is an indicator of proper mental development as well as the sequencing of task difficulty).

Every task could be completed by the child on one of five levels, from refusing to complete the task to solving it on a level demonstrating a mathematics aptitude (manifesting the aforementioned mental abilities). The research was continued to be carried out until the merit and statistical analyses of the kids' behaviour allowed for an assessment of their competences on a five-grade scale. 124 older kindergartners and primary school students took part in this laborious study at first. The number then rose to 487 [13].

It is worth mentioning that diagnostic tasks differ from conventional tests [14] in such a way that the child is able to complete the tasks on different levels, on par with knowledge and abilities. A *good-bad* grading system is not used, as every task has specific levels of completion. The researcher observes and analyses the behaviour of the child and chooses a level appropriate for the child. It is then compared to the age and

13. In order to observe the individual differences in the competences of the children, I have not singled out any individual students from all of the kindergartens, as well as pre-first grade and first grade classes taking part in the study. From my experience, if a teacher is asked to single out a number of children to take part in a study, they will choose gifted children. Even when they are asked to pick weaker students. What's more, in order to carry out research, it is necessary to obtain the permission of: the principal of the educational institution, the teachers of the children in question, and all of the parents of the children as well as the kids themselves, one by one. It seems just, even though it very much complicates carrying out research programs and reduces the number of children eligible to take part.

14. More information regarding the differences can be found in the E. Gruszczyk-Kolczyńska & E. Zielińska book *Nauczycielska diagnoza edukacji matematycznej dzieci. Metody, interpretacje i wnioski* [*A teacher's diagnosis of the mathematics education of children. Methods, interpretation, and conclusions*], Wydawnictwo Nowa Era, Warsaw 2013, chapter 1.

the educational situation of the child, which is graded according to the grading guide.

If the diagnostic tasks form a series, the learning phase can be incorporated, e.g. guiding the child step by step towards solving the task. Creating a similar task can also be proposed to the child, as well as presenting ill-constructed tasks to see if the child will notice. I have made use of these suggestions while creating the diagnostic tasks for each of the aforementioned areas of mathematical activity. The tasks were presented to the child in such a way that they could refuse to solve them, make use of the learning phase (in various ways), solve the tasks on their own, create similar tasks for the researcher, and then observe and check whether the researcher is making mistakes while solving the tasks. This way, the child had the opportunity to exhibit each of the aforementioned mental attributes of the mathematically gifted.

This stage of the study resulted in the formulation of diagnostic tools for the recognition of mathematics aptitude in older kindergartners and primary school students [15]. The next step was estimating the number of mathematically gifted children. A number of 182 older kindergartners and primary school students took part in this stage of the study.

The merit and statistical analyses of the results of the research allowed for the indication of children with lower competence (less knowledge and abilities than their peers). The remaining children exhibited varying levels of competence: they had average skills in some of the aspects of maths activity, while surpassing their peers in other aspects. It was possible to discern mathematically gifted and highly gifted children in the varying levels of competence group. All of the groups will be described.

Children with less knowledge and abilities than their peers

This group contained poorly performing children. They refused to solve the simplest diagnostic tasks of several of the areas of maths activity, even when the researcher presented the tasks again and tried guiding the children step by step towards solving the tasks. There were also children who did a bit better. They did not refuse to try solving the tasks,

15. The tools are presented (with research aids) in the aforementioned report from the realisation of the R1700603 project, *Wiadomości i umiejętności oraz zarysowujące się uzdolnienia matematyczne starszych przedszkolaków i małych uczniów.Podręcznik...* What's more, after carrying out this program, I have created tools for the diagnostic recognition of the mathematics aptitude of children, adapted for the needs and possibilities of teacher diagnosis. They are described further in this work, and presented in their entirety in the book *O dzieciach matematycznie uzdolnionych...*, part 2, as well as in the E. Gruszczyk-Kolczyńska & E. Zielińska book *Nauczycielska diagnoza edukacji matematycznej dzieci...*, chapter 7.

though their involvement solely included helping the researcher by performing simple tasks, such as handing blocks. Some children were a bit more conscious during the learning phase, e.g. they repeated what the researcher showed them. Unfortunately, this approach to the learning phase was insufficient for them to solve the next task of the series on their own, as it was more difficult. About a third of the children taking part in the study used this approach. The number of boys and girls in this group was more or less even.

Children with varying competence and children mathematically gifted

According to the performed research, circa 2/3 of the children exhibit varying levels of competence (reasoning, information and mathematics skills) in a given area of mathematics education. If a child performed similarly to their peers in a given series of diagnostic tasks, they performed on a higher or lower level in another series. As this was independent of the age and sex of the children taking part in the study, the cause must be the mathematics education carried out at home, at kindergarten and at school [16]. I defined a child to be mathematically gifted if they fulfil the given criteria in at least one area of mathematics activity:

- Makes use of the learning phase to apply what has been learned while solving tasks;
- Has the sense and critical thinking ability to notice the absurdities in ill-constructed tasks;
- Notices that the task is being solved incorrectly (by the researcher, making mistakes on purpose);
- Creates maths tasks on their own, showing a creative mindset regarding maths activity;
- Is not bored while solving series of maths tasks.

I assumed that if a child is able to exhibit all of the aforementioned mental abilities in one area of maths activity, they can probably do so in another. It is hard to imagine e.g. Chris having a sense of meaning behind the ability to count and not having it in regards to calculating, measuring, etc. Keeping that in mind, I assumed that **an indication of the outline of mathematics aptitude of a child is that they show**

16. Among the areas in which a lot of the children had embarrassingly little knowledge and skills are: *the value of money and tasks regarding buying and selling, measuring length, mass, capacity, and time, as well as tasks requiring the aforementioned skills*. The reason is that the adults – teachers and parents – do not pay enough attention to these areas while teaching. The most favoured areas of mathematical activities are: *spatial awareness, counting, addition and subtraction*.

high levels of competence in at least one area of mathematics activities, as this is typical for mathematics aptitude. Studies suggest that more than half of the children taking part in the study fulfil this criteria. This confirmed the earlier research assessment regarding the school performance of the children taking part in the *Dziecięca matematyka* [*Children's mathematics*] program. The group of mathematically gifted children also contains highly gifted children, who are described further.

Mathematically highly gifted children

The group of gifted children consisted of those who showed high levels of competence in as many as 10 out of 13 analysed areas of maths activity. There were even two kids (out of 41) in the group of four-year-olds who exhibited high levels of competence in five and six areas of maths activity respectively, matching the competence levels of significantly older children. I concluded that if children are able to exhibit such skills at four years of age, it is probably also possible at five, six, and seven years of age.

Which is why I assumed that **if a child exhibits high levels of competence (reasoning, information, and skills) in give or more areas of mathematics activity, they can be called mathematically highly gifted**. What's more, high levels of competence also include the manifestations of the gifted kids' mental abilities.

How many older kindergartners and primary school students are mathematically highly gifted? The studies show that:

– The outlines of mathematics aptitude can be observed in four-year-old children. This confirms the thesis statement concerning the manifestation of high levels of competence[17]. The problem is that parents and kindergarten teachers cannot fathom the fact that four-year-old children can manifest high levels of mathematics aptitude. Which is why it the aptitude is rarely cared for, which harms the mental development of the children.
– **Outlines of high levels of mathematics aptitude are visible in five-year-old children. Based on the research I estimate that every fifth five-year-old child manifests their high levels of mathematics aptitude**. Despite this, teachers are convinced that five-year-olds are simply not good at maths, which

17. This is confirmed by H. Gardner (*Inteligencje wielorakie. Nowe horyzonty w teorii i praktyce* [*Multiple Intelligences: New Horizons in Theory and Practice*], Wydawnictwo Laurum, Warsaw 2009) in his characterisation of intelligences, including logical-mathematical intelligence.

is why mathematics aptitude in this age group seems impossible to them. They are also wrong in the assessment of the mental abilities of children, even those with high levels of mathematics aptitude;

– **Six years of age is the optimal time for mathematics aptitude to manifest. Mathematics aptitude is clearly defined at this point, which is why I estimate every fourth child to be highly gifted**. This applies to children attending kindergarten or pre-first grade classes at school in Poland.
– **Primary school students manifest their mathematics aptitude significantly less often. Even though before enrolling to primary school every fourth child can be considered highly gifted, after a few months of primary school this changes to every eighth student**. What's more, an analysis of the functioning of the primary school students solving the series of diagnostic tasks showed that they are less critical and less brave in creating tasks on their own. They expect help in solving tasks more often and less often react to errors in tasks. It is worth noting that the research was carried out in April, which is the eighth month of a school year.

Why, after just a few months of being a first grader, does the number of children manifesting their mathematics aptitude drop significantly?

To explain this, I have observed mathematically gifted children at kindergarten and at school in mathematics education classes [18]. I determined that: the reason that the number of young students manifesting their mathematics aptitude drops significantly is the emphasis on socialisation, which is a part of school education. This is the side effect of the teacher showing the kids how to behave in a group of students from day one [19]. In turn, the children do their best to fit into the model given by the teacher, as the teacher is the most important person for them at

18. In the analysis of the process of educating children, I drew from my earlier experiences gained during the assessment of the educational value of the *Dziecięca matematyka* [*Children's mathematics*] program, a well as during the research on the reasons of specific hardships in learning maths.

19. Everyone has to volunteer to answer in a given way, perform tasks according to the given model, speak to the teacher according to the given model, answer questions the way they're supposed to, use their textbooks the same way as others do, etc. This helps the teacher educate every child the same things (by sticking to the syllabus) in the same way (the kids all solve the same tasks in their textbooks) at the same time (as is required by the way classes are organised by the school).

this point. The problem is that the model is an average student. There is nothing wrong with socialisation itself. The faster the children understand the concept of a student, the less trouble there is with their behaviour and education. There is a dangerous downside to this, as the socialisation extends to the mental functioning of the young students, enforcing, mediocrity [20].

The problem is that mathematically gifted children have immense problems accommodating to the model of the average student due to their mental abilities and the scope of information and skills they have. Because:

- They ask too many inquisitive questions, insisting on getting the answers from the teacher, with little regard for the other children;
- They have more knowledge and skills than their peers, which makes them bother the other children and the teacher out of boredom;
- They do not yet know how to refrain from critical remarks when maths tasks are trivial or badly constructed, etc.

No wonder they are constantly chastened, scolded, and called to order. This quickens the process of shaping them into mediocrity. What's more, the parents of gifted children rarely take the side of their child, as they usually support the teacher's ideas. They force their child to be obliging at school, which makes the child average. Such strong pressure makes mathematically gifted children understand some "life lessons" after a few months of attending first grade. They begin to understand that there is no point in:

- Solving maths tasks quickly, as they have to wait for all the other children to finish anyway;

20. Cz. Kupisiewicz (*„Zmiana" (change) i „wzmacnianie" (strengthening) – słowa-klucze współczesnych reform szkolnych* [*„Change" and „strengthening" – the keywords of modern school reforms*], in: *Edukacja narodowym priorytetem. Księga jubileuszowa w 85 rocznicę urodzin Profesora Czesława Kupisiewicza* [*Education as a national priority. A Festschrift for the 85th birthday of Professor Czesław Kupisiewicz*], Wydawnictwo Wyższej Szkoły Humanitas, Sosnowiec 2009) states that the plague of mediocrity encompasses multiple education systems across the world, so it's not just a Polish thing. The plague of mediocrity is defined by Cz. Kupisiewicz as *. . . przeciętne szkoły stosują metody i formy organizacyjne nauczania i wychowania nastawione na przeciętnych uczniów, którym stawiają przeciętne na ogół wymagania, a zaniedbują dzieci i młodzież o ponad przeciętnych możliwościach i uzdolnieniach marnotrawiąc bezcenny „kapitał ludzki" o ogromnym znaczeniu dla przyszłości kraju. . .* [*. . . average schools use methods and organisational forms of educating aimed towards average students, of whom average things are demanded, at the same time neglecting the children and the adolescents who have above-average capabilities and aptitude, wasting a priceless human capital that is of enormous significance for the future of the country. . .*]

- Seek a better, more thought-out way of solving a task, as only the so-called proper way is the only way of solving a task;
- Exhibiting a higher level of maths skills, as such behaviour is treated as boasting, which is unacceptable in the eyes of their peers;
- Be critical of the given tasks, even if the tasks are absurd, and the way of solving them is incorrect. Such a "deviation" is not acceptable by the teacher's standards.

The gifted children experience this week after week, month after month. No wonder all the socially sensitive kids stop exhibiting their mental capabilities after a few months of attendance, and, what's worse, making use of them. This is why there is less and less mathematically gifted children from one grade to the next. What it all comes down to is that in later grades there are up to two children who can gain information and skills easily enough to be considered mathematically gifted.

What teachers think of mathematically gifted children and why they are wrong in assessing their mental capabilities

The way a child manifests their mental capabilities depends on the teacher. Whether they will eagerly polish their maths skills and happily use them in their maths activities, or get discouraged towards everything related to calculating and measuring. If the teacher believes in the child's mental capabilities, he will not be let down. And vice versa – if the child is being underestimated by the teacher, the child will underperform. As well as have a lot of behaviour issues.

Which is why, during the course of the research regarding the mathematics aptitude of older kindergartners and primary school students, I have asked the teachers to evaluate the mental capabilities of each child taking part in the study, using a very simple scale:

- The child is distinctly younger, performs poorly, more childish than their peers;
- Functions on the same level as the other children, so they're on an average level;
- The child stands out in terms of mental capabilities.

The results of this study show a **surprisingly strong tendency of the mental capabilities of mathematically gifted children being undervalued by the teachers who educate them**. For instance, in a group of 8 mathematically gifted five-year-olds, only two were graded as standing out in terms of intellect. In a group of seven-year-olds, only every fourth child was labelled as such. What's more, this grade was given to children who outperformed their peers in more than five (out of 13) areas of maths activity, exhibiting the mental abilities described earlier. There are most probably a few reasons as to why the mental capabilities of mathematically gifted children are undervalued.

It is likely that the behaviour problems exhibited by the gifted children obscures their mental abilities in the eyes of the teacher. The kids, being bored, cause trouble during lessons and do what they're not supposed to. They're know-it-alls who ask lots of questions, often unrelated to the topic of the lesson. They criticise trivial maths tasks or propose alternative ways of solving tasks. When scolded, they rebel by disregarding the rules, etc. This is probably why the teachers do not remain objective, as they think misbehaving children cannot be gifted. Definitely not in mathematics!

Another reason might be the fact that all the soon-to-be teachers taking part in academic courses relating to kindergarten/primary school education in Poland do not have any classes [21] preparing them for the recognition of the aptitude of children and adjusting their education accordingly.

Critical (sensitive) periods of the development of mathematics aptitude in older kindergartners and students

The critical periods of the development of mathematics aptitude are kindergarten and primary school years, during which the fate of mathematically gifted children, as well as younger and older students is decided. Psychologists use the term *critical period* in regards to pairing the learning process to the mental capabilities of a child, including the effectiveness of obtaining life skills [22]. This concerns a period of enhanced

21. An analysis of the timetables of academic courses for teaching has shown that in the academic year of 2011/2012, not one of the names of the classes suggested that any kind of discussion regarding gifted children will be taking place during the course of the year. There were, however, several classes preparing for work with children with mental development problems, as well as children with insufficient school maturity, etc.

22. M. Przetacznikowa (*Podstawy rozwoju psychicznego dzieci i młodzieży* [*The basics of the mental development of children and adolescents*], WSiP. Warsaw 1978, chapter *Problem okresów krytycznych w ontogenezie* [*The problem of critical periods in the ontogeny*]) states that psychologists use the word sensitive when describing critical periods, defining them as such: *okresy krytyczne uważa się za sensytywne, nie wykluczając możliwości przyswojenia sobie przez jednostkę określonych sprawności i umiejętności także poza granicami czasowymi fazy krytycznej, optymalnej dla ukształtowania danej funkcji, jakkolwiek przyjmuje się, że proces uczenia się jest wówczas mniej skuteczny* [*critical periods are considered sensitive, not excluding the possibility of the individual learning certain capabilities and skills outside the time frame of*

susceptibility to the learning process, as shown by the optimal readiness of the central nervous system to form what psychologists call schemas or mental representations [23]. Teachers call them information and skills. This optimal readiness for learning is called by L. S. Vygotsky the *zone of proximal development*, underlining its importance in the dynamics of mental development and the effectiveness of education [24]. The point of this term can be described as such:

- Children create cognitive schemas in their minds drawing on specific logical experiences due to the interiorisation mechanism [25].
- When a parent organises the child's learning process in such a way as to allow the child to gather the logical experiences needed to create schemas, it eases the interiorisation and increases the tempo of mental development [26];
- The child uses schemas in acquiring information and skills, which fulfils their developmental needs as well as reinforcing their self-esteem and agency.

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the critical period which is considered optimal to obtain such functions, although it is assumed that the learning process is then less effective].

23. These schemas are named in different ways: J. S. Bruner (*Poza dostarczone informacje* [*Beyond the presented information*], PWN, Warsaw 1974, part 4 Procesy reprezentacji w dzieciństwie [Processes of representation in childhood]) uses the term representations, and J. Piaget (*The Psychology of the Child*, PWN, Warsaw 1966, as well as *The Equilibration of Cognitive Structures: The Central Problem of Intellectual Development*, PWN, Warsaw 1981) calls them cognitive schemas (structures), explaining that they are created by assimilation and accomodation and their mutual coordination and equilibration.

24. It is a theoretical construct created by L. S. Vygotsky (*Wybrane prace psychologiczne* [*Selected psychological works*], PWN, Warsaw 1971, pp. 351-365, 517-530) to create a relation beneficial for the mental development of children between the maturing of the central nervous system and the course of the learning process, which is organised by their parents.

25. I describe the mechanism of interiorisation (learning through internalising experiences) in detail in the cited work *Dzieci ze specyficznymi trudnościami w uczeniu się matematyki...*, chapter *Teoretyczne podstawy zajęć korekcyjno-wyrównawczych* [*The theoretical basis of compensatory activities*].

26. A more in-depth explanation of such educational opportunities can be found in the work of E. Grusczyk-Kolczyńska and E. Zielińska *Zajęcia dydaktyczno-wyrównawcze dla dzieci, które rozpoczną naukę w szkole* [*Compensatory activities for children beginning school education*]. Wydawnictwo Edukacja Polska, Warsaw 2009, pp. 24-43. It is also the topic of a lecture by E. Gruszczyk-Kolczyńska from the *O dobrym wychowaniu przedszkolaków* [*About the proper upbringing of kindergartners*] series called *O wspomaganiu rozwoju umysłowego dzieci* [*Supporting the mental development of children*], "Bliżej Przedszkola" 2010, issue 9.

Learning does not give any desirable effects if the parent is trying to facilitate a skill which the child is not yet ready to master. For instance, if the child can perform addition and subtraction of objects by counting them after every move, they won't be able to do it from memory. The parent cannot change that in any way. They will achieve a desirable effect after helping the child understand counting using e.g. the child's fingers. If this method of counting no longer causes problems for the child, they need to be supported in their transition to calculating from memory. This is how the zone of the development of a child's calculating skills progresses.

The results of the research presented earlier shows that the earliest such interval occurs during the last year of kindergarten and the beginning of school education. That is when children exhibit and develop the mental traits responsible for mathematics aptitude. Kids can be assisted with this process, but they can also be suppressed. All it takes is to disregard the child's maths activity, decrease the amount of mental work of the child by cutting back the amount of calculating, etc. If the outlining maths aptitude is not developed and nurtured, the aptitude wastes away, which cannot be made up for in the following years of education. This is confirmed by the research presented earlier regarding the school fate of mathematically gifted children.

Another critical period occurs when children begin fourth grade in Poland, which is when their maths education starts being conducted by a maths teacher. This occurs because there is a significant shift in the way of the way information and maths skills are formed. These changes are often very significant and emotionally draining even for gifted children, so much so that a lot of them lose faith in their own mental capabilities, therefore losing motivation to learn maths. This, unfortunately, happens often.

This happens due to maths teachers overrating the mental capabilities of students. When assembling the contents and methods of maths education it is assumed that the students have the ability of formal operational thinking, swiftly making use of symbols [27]. One would assume gifted students would rise to the challenge due to their superior development of operational thinking when compared to their peers [28]. Unfor-

27. Cf. Siwek H., *Dydaktyka matematyki. Teoria i zastosowania w matematyce szkolnej* [*Didactics of mathematics. Theory and application in school mathematics*], WSiP, Warsaw 2005, chapters *Klasy pojęciowe w podręcznikach klas IV-IV* [*Conceptial categories in textbooks for Polish grades 4-6*], *Elementy logiki w matematyce szkolnej* [*Elements of logic in school mathematics*], *O odkrywaniu, formułowaniu i dowodzeniu twierdzeń* [*Of the discovery, formulation, and proof of theorems*].

28. It is worth noting that J. Piaget was a bit too optimistic in stating that achiev-

tunately there are also many difficulties that need to be faced, and this blocks the development of mathematics aptitude.

Operational reasoning on the formal level, similarly to the level of concrete operations [29], gradually expands in a way resembling circles in water, absorbing other zones of intellectual functioning. Just because students can make use of formal operational reasoning in the context of specific areas of maths does not mean that this happens for all areas. Which is why a lot depends on the maths teacher and whether they know how to:

– match the formulation of information and skills to the actual mental capabilities of the students in concrete contexts;
– understand the mental characteristics of a gifted student, relish their accomplishments and encourage them towards independent maths activity;
– support the gifted student in the pursuit of understanding new available areas of maths, as well as allow them showing off their abilities on contests and competitions.

It is worth noting that fulfiling these quite standard expectations is hard, as maths teachers are seldom taught during courses how to properly take care of gifted students. They learn a bit about the proper mental development of students in didactics of mathematics classes [30]. If the

ing this level of reasoning occurs at 12 or 13 years of age (*Studia z psychologii dziecka* [*The Psychology of the Child*], PWN, Warsaw 1966). More recent British studies show that 80% of students are still not at the early level of the concrete operational stage at 14 years of age. The basis of the research were studies conducted on 10,000 primary school students. Cf. M. Shayer, D. E. Küchemann, H. Wylman *Distribution of Piagetian Stages of Thinking in British Middle and Secondary School Children* „British Journal of Education Psychology" 46, 1976 .

29. More information available in the book]textitWspomaganie dzieci w coraz precyzyjniejszym klasyfikowaniu. Stosowanie klasyfikacji w edukacji matematycznej [Assisting children in more precise classification. Use of classification in mathematics education], in: *Wspomaganie rozwoju umysłowego oraz edukacja matematyczna dzieci w ostatnim roku wychowania przedszkolnego i w pierwszym roku szkolnej edukacji. Cele i treści kształcenia, podstawy psychologiczne i pedagogiczne oraz wskazówki do prowadzenia zajęć z dziećmi w domu, w przedszkolu i w szkole* [*Assisting in mental development and the mathematics education of children in the last year of kindergarten and the first year of primary school. Goals and instructions for teaching, the psychological and pedagogical basis, and tips for conducting classes with children at home, at kindergarten, and at school*], ed. E. Gruszczyk-Kolczyńska, Wydawnictwo Edukacja Polska, Warsaw 2009, chapters 7, 8, 10, 11, 12, 17, 19, 20, 21, 22.

30. This is discussed only in specialised psychological works, which are not well known even among psychologists, as they were issued over half a century ago in limited circulation in Poland. For instance, H. Aebli *Dydaktyka psychologiczna. Zastosowanie*

curriculum of educating maths teachers changes, their students will have a chance of developing their mathematics aptitude. Unfortunately, for now this is only possible for students who stumble upon exceptional teachers during their education.

The next critical periods occur as students finish a stage of education and move onto the next. A common feature of these critical periods is that less attention is being paid towards scaling the level of material in maths education to the mental capabilities of students. The accompanying coincidental situations in family life, kindergarten education, and school education yield either positive or terrible results.

Can an adult without a mathematics education effectively develop the mathematics aptitude of children?

Explaining this issue is necessary, seeing as during the first critical period of developing maths aptitude, the people in charge of raising and educating the children are their parents, kindergarten teachers, and non-specialised primary school teachers. Only some of them are interested in maths activity, and their education varies.

Before working on the program regarding assistance for the growth of mathematically gifted children [31], I have asked maths didacticians this: *What has to be done in order for a gifted child's mind to grow?* Everyone exclaimed that the tasks the children are given have to be interesting. This is probably why books such as *I ty zostaniesz matematykiem* [*Even you can be a mathematician*] are filled with such tasks. However, from my experience, I can state that <u>*feeding* interesting maths tasks to gifted children is a pedagogical trap</u>. It causes a type of emotional addiction caused by the following events:

– The joy of solving interesting tasks compels the children to demand more and more tasks;
– The children understand that an adult is the creator of the tasks at hand and experience the adult's intellectual superiority in maths activity;
– Involuntarily they ascertain that their skills are very poor, which is why they abstain from creating maths tasks on their own.

teorii Piageta do dydaktyki [*Psychological didactics. The application of Piaget's psychology to didactics*] (PWN, Warsaw 1959, reissued in 1982), B. Inhelder and J. Piaget *The Growth of Logical Thinking From Childhood to Adolescence* (PWN, Warsaw 1970).

31. This program was created after the aforementioned research was carried out. It is included with psychological and pedagogical commentary in the cited work *O dzieciach matematycznie uzdolnionych...* in the chapters of part four.

This also results in the fact that children are less and less interested in independent maths activity in real-life situations. This effectively blocks the development of the maths aptitude of children.

This is why in my concept of assisting gifted children[32] there are important series of tasks which are assembled and solved in an alternating fashion, like so: the adult creates an easy task for the child to solve, followed by the child creating a similar task for the adult, who solves the task. This is done several times. Among alternating between assembling and solving the tasks, there is also time for error correction: the adult intentionally makes a mistake while solving one of the tasks created by the child in such a way so that the child can notice the mistake. The child then corrects the mistake, which makes them understand that making mistakes is not a tragedy, they just need to be corrected. This encourages the children and helps them have a sense of meaning, as well as teaching them how to behave when they make a mistake. Which is also what this is about. Children can work in pairs in a similar manner: one of them creates a task, the other one solves it. And vice versa.

This method yields positive educational results as long as the tasks created by the adult at the beginning of each series are simple. The child has a feeling of intellectual partnership in such a situation. The child begins an intellectual pursuit, by wanting to impress the adult with their skills, whereas the adult can keep track of how far can the child's reasoning go. During such an intellectual skirmish, the child:

– forms a creative approach towards maths activity;
– develops a sense of meaning and the ability of making an intellectual effort;
– improves their maths skills;
– feels productive and improves their self-esteem.

This is why it is better to support the children in independent maths work rather than burying them in interesting tasks. Though this does not mean such tasks are never to be used. They are a source of logical and mathematical experience for the children.

There is also no reason for the adult to be afraid that their lack of maths competences will hinder the child's accomplishments. It is the child who is set to become extraordinarily successful in learning maths, not the adult. It works similarly in sports — the coaches of gifted athletes are proud of their results and do not feel discouraged by the fact that the athletes are achieving more than the coaches possibly could.

32. This is described in chapters three and four in the cited work *O dzieciach matematycznie uzdolnionych...*

Arguments for the recognition of the mathematics aptitude of older kindergartners

I have been asked multiple times, *Is it not too early to facilitate the development the mathematics aptitude of kindergartners? Will it not restrict the mental capabilities of the children? Will it not harm their other interests?* etc. This is not something to be afraid of. Five, six, and seven years of age is a great time to facilitate the development of maths aptitude. This is when the first critical period of developing maths aptitude takes place in the child's life. If parents will not properly facilitate the development of their children's aptitude, it will not be manifested. It is hard, if not impossible, to make up for the lost time in the following years. This is another strong argument for the early recognition of the mathematics aptitude of children.

In the kindergarten years, as well as the first years of primary school, the child develops traits that serve as the basis for achieving extraordinary results, not only in mathematics education. Plus, there is no reason not to facilitate the development of any other interests the child might have alongside their mathematics aptitude.

If, for instance, the parents make sure that the child takes part in the fascinating shows at the Copernicus Science Centre, the child will want to become a space expert. If the child is encouraged to perform experiments and get to know the complexity of physical phenomena, then this is what the child will be fascinated with. If a grandfather and grandson e.g. disassemble a motorcycle together, and then assemble it back, the child will become fascinated with mechanics. In order to succeed in these and similar branches of activities, high levels of mathematics competence are required. There will be time for profiling and expanding this aptitude onto other branches of science.

Let us focus on recognising aptitude, as supporting children in developing their aptitude can only begin once the aptitude is diagnostically recognised. The realisation of the project *Rozpoznawanie i wspomaganie rozwoju uzdolnień do uczenia się matematyki u starszych przedszkolaków i małych uczniów* [*Recognising and encouraging the development of aptitude for learning mathematics in older kindergartners and primary school students*] ended with the formulation of the concept of assessment and of diagnostic tools designed for the purpose of this research [33]. The prob-

33. The report regarding this research (named R1700603) contains the assumptions of the diagnosis, the research procedure, as well as tools for use in the study along with a statistical confirmation of their diagnosticity. A program can be created, on the basis of the diagnosis, for the assistance of mental development and custom-made mathematics education, matching the development and educational needs of a gifted child.

lem is that such an assessment takes time, is carried out individually and requires essential prep work.

One more concept and set of diagnostic tools intended for teachers had to be created. They are the first adults who can notice and with relative objectivity assess the mathematics aptitude of children. This, in turn, allows for a head-start in taking proper pedagogical care of the children and the development of their extraordinary gift.

Short description of the teacher diagnosis for the recognition of mathematics aptitude in older kindergartners

After finishing the aforementioned research, I have worked for months on such a concept of a diagnosis and set of diagnostic tools that would be tailored to the specifics of a teacher's diagnosis. In the spring of 2011, the diagnosis was tested in realistic conditions of kindergarten teaching: in 20 kindergartens in Olsztyn.A number of 731 children took part in the study [34]. The concept of the teacher diagnosis is presented in-depth in the cited work *O dzieciach matematycznie uzdolnionych...* [35] along with diagnostic tools, hints regarding interpretation and further pedagogical work [36].

The goal of the teacher diagnosis is to assess what the children know and can do in given areas of mathematics activity, and to deduce their mathematics aptitude. It encompasses children who are nearing the end of kindergarten education as they manifest their mathematics aptitude, the first months of primary school education the latest. The diagnosis consists of two segments. The first segment is the screening study [37], the

34. The results of this study were presented on the conference *Dziecko uzdolnione matematycznie – diagnozowanie oraz wspieranie w rozwoju i edukacji* [*Mathematically gifted child — diagnosis and support in development and education*] (19 September 2011, the In-Service Teacher Training Centre in Olsztyn) in the works of E. Zielińska *Rozpoznawanie uzdolnień matematycznych dzieci* [*Recognising the mathematics aptitude of children*] and J. Jastrzębska *Matematyczne uzdolnienia dzieci w olsztyńskich przedszkolach – problemy, wyniki badań, wnioski* [*Mathematics aptitude in Olsztyn kindergarten children – problems, research results, conclusions*].

35. The chapters of the second part of the book focus on this subject.

36. All wishing to adhere to these resources have to prepare, learn the research scenarios, gather the necessary items, and understand the notation of the results and interpretation. Based on the feedback, this does not seem to be a difficult endeavour.

37. The screening study, apart from the goal of the diagnosis, allows for the realisation of a second goal of circling out the children who lag behind their peers, in order to assemble remediation activities for them. This will better prepare the children for primary school education.

goal of which is to assess the individual differences in given areas of maths activity and to pick out the children who either:

– perform worse than their peers;
– have skills and information on an average level;
– or stand out with their level of skills and information.

The second segment of the teacher diagnosis is individual research of the children. If the goal of the teacher diagnosis is the assessment of mathematics aptitude, this segment will encompass all the students that performed exceptionally in the first segment of the diagnosis. The two segments of the diagnosis are to be carried out in order. In the first segment, the diagnostic goals are combined with the curriculum content of maths education at kindergarten and at school. Which means that this segment can be carried out in mathematics classes, in the morning. The second segment has to be carried out at another time, as it involves individual research.

The leading method in the teacher diagnosis for the recognition of mathematics aptitude in children are *diagnostic experiments*. Each of the experiments contains:

– Organisational work to ensure that the conditions for every child are as identical as possible: gathering the necessary number of items for manipulation and preparing the assigned space for every child;
– Series of diagnostic tasks created in such a way that every child will understand what is expected of them and what is to be done;
– A uniform approach to the observation and interpretation of the activity of the children solving the diagnostic tasks. This also includes uniformity in logging the results of the studies on the diagnostic sheets.

The *series of diagnostic tasks* that the child is to solve are important in every such experiment. I based the tasks on: a) the correctness of the development of Piaget's concrete operational thinking, b) reasoning by insight, as children prefer to do this while solving maths tasks, c) the models of development of mathematics aptitude related to calculating [38] as well as the scope of the curriculum content in kindergarten and primary school education [39]. I have also taken into consideration findings regarding

38. This is discussed in E. Gruszczyk-Kolczyńska and E. Zielińska *Liczenie. Wspomaganie dzieci w ustalaniu prawidłowości, które są stosowane w liczeniu obiektów. Kształtowanie umiejętności liczenia* [*Calculating. Assisting children in determining the regularities of counting objects. Developing numeracy*], in: *Wspomaganie rozwoju umysłowego oraz edukacja matematyczna dzieci w ostatnim roku wychowania przedszkolnego i w pierwszym roku szkolnej edukacji...*

39. This is described in the core curriculum for early primary school and kinder-

the way children undertake and solve instructions and tasks: the attitude towards the tasks, understanding their meaning, solving the tasks in the proper order, feeling satisfied after solving the tasks.

The teacher diagnosis focused on these areas of mathematics education in which the children show off their knowledge and skills, as well as the mental attributes indicating mathematics aptitude. Unfortunately, for some reason some of the teachers and parents do not pay enough attention to some of the areas of mathematics activity. Usually those regarding measuring length, capacity, mass, and time. Children are also sometimes not given pocket money, and this causes them to neither understand the value of money, master money-related calculations, nor understand any activities related to economics [40]. The child not solving diagnostic tasks relating to this matter might stem from educational neglect, and not from a subpar mental performance.

Having this and other complications in mind, I have established three areas of the mathematics education of children. They are:

- ✓ counting;
- ✓ addition and subtraction, including window-related tasks;
- ✓ creating and solving instruction-based tasks, including tasks with intentional errors.

This is backed by organisational and substantive arguments. By counting and calculating, children make use of intellectual activities that entail success in basically all areas of maths activity carried out at home, at kindergarten, and at school. Also, the details of the development of numeracy in children is well-known, which allows for a relatively precise determination of whether the knowledge and abilities of a child are adequate to their age, or whether they are under- or over-performing. While creating and solving tasks related to counting and calculating, children can exhibit mental characteristics consistent with mathematics aptitude. By encountering the intentionally placed mistakes while solving tasks, as well as misconstructed instruction-based tasks, the children can exhibit a sense of meaning. Diagnostic tools requiring the use of counting and calculating can be created in such a way so that the children could solve them in group research (the first segment of the diagnosis) as well as

garten education, attachments 1 and 2 of the 23 December 2008 regulation implemented by the Ministry of National Education in Poland.

40. The results of research regarding economics education are presented by M. Kupisiewicz in the book *Jak kształtuje się u dzieci rozumienie wartości pieniądza. Z badań nad rozumieniem wartości pieniądza i obliczeniami pieniężnymi.* [*On children understanding the value of money. Research on the understanding of the value of money and money-related calculations.*] Wydawnictwo APS, Warsaw 2004.

individual research (the second segment of the diagnosis). The execution of the diagnostic tasks in the area of counting and calculating does not take long.

A short description of the first segment of teacher diagnosis

The research is to be carried out in the morning, as part of mathematics classes. Which is why this segment was created in such a way as to permit the simultaneous realisation of:

- Educational goals: improving counting and calculating skills in the children taking part in the diagnosis (in accordance to the recommendations of the new kindergarten core curriculum);
- Diagnostic goals: assessing the level of knowledge and abilities of the children in regards to counting and calculating and choosing those who are: a) average, b) performing poorly as compared to their peers, c) perform better than their peers – these children move on to the second segment of the teacher diagnosis.

The first segment of the teacher diagnosis acts as a screening study and consists of two diagnostic experiments: *Counting and Addition and subtraction*. Every experiment contains *introductory tasks* and *series of diagnostic tasks*. Why the *introductory tasks*? They are necessary if the *diagnostic tasks* are to be solved in a group[41], because:

- They direct the reasoning of the children towards the intellectual activities used in the *series of diagnostic tasks*;
- They will familiarise the children with the words and instructions used by the teacher in the series of diagnostic tasks, which will make them better understand the meaning of the diagnostic tasks.

There is one more solid argument for the inclusion of *introductory tasks*. That is the possibility of assessing the ease of acquiring information and maths skills, which is a mental trait of gifted children. This can be assessed based on how the child makes use of the logical experience gathered during the *introductory tasks*. Because these are situations which contain the experience that is the *building block* of perfecting the skills of counting and calculating. If the child can acquire the experience while solving diagnostic tasks, this indicates an ease of acquiring knowledge and

41. These tasks are usually solved during individual diagnosis. If the child's behaviour suggests that *they are lost*, the researcher can repeat the instructions or the whole task, use gestured to point out the important part, encourage the child towards being more active, etc. If the *diagnostic task* is to be solved simultaneously by all of the children in the group – which is what happens in the first segment of the diagnosis – these forms of encouragement and support cannot be employed. This can be remedied by properly incorporating *introductory tasks*.

maths skills, which is an important indicator of mathematics aptitude. In order for all this to happen, *introductory tasks* must be carried out before the *series of diagnostic tasks* are solved.

In order to fulfil diagnostic standards, the first segment of the diagnosis is in the form of scenarios used as lesson plans in order to conduct lessons with children and fulfil the goals of the diagnostic research at the same time. The scenarios contain in-depth guidelines for:

– The conditions in which the children are to solve the *introductory tasks* and the *series of diagnostic tasks*. The point is that the same items should be used by the children and the same amount of time should be available for everyone to solve the tasks.
– Familiarising the children with what they are to do in order to properly solve the *introductory tasks* and the *diagnostic tasks* as well as observing the children while they are solving the tasks;
– The criteria for the grading of the performance of the children solving *diagnostic tasks* created in such a way so that every child can be assessed individually, but also compared to all the other children taking part in the diagnosis.

The scenarios also contain the descriptions of the tiers the children can be categorised under based on the results of the *diagnostic tasks*. By observing the children and having the descriptions in mind, it can be assessed whether the tasks were: a) too hard for them (low tier), b) hard, but doable (average tier), c) easy (high tier). The last part of the scenario of the first segment of the diagnosis contains tips for the logging of the grades of the children taking part in the study, as well as the rules of interpreting the results of the study and conclusions allowing for the improvement of the course of education for the children.

Remarks regarding the interpretation

The situation of the children for whom the diagnostic tasks were too much for their mental capabilities is troublesome (most or all tasks done poorly). Also, these children do not make use of the introductory tasks for the following reasons:

– They have a lower susceptibility towards the learning process that is being facilitated by the researcher, which is why the experience gathered while carrying out the introductory tasks is not enough for them to understand and solve the diagnostic tasks;
– They have trouble concentrating and cannot comprehend the complexity of the tasks, which is why they do not follow instructions unless helped by an adult.

The scope of experiences gathered during their kindergarten and primary school classes is not enough for these children to develop mathematics aptitude, mostly in regards to counting and calculating. If these children are to benefit from mathematics education, they have to take part in remediation activities.

The children graded as average also need support, but for a different reason. In spite of having issues, they can still solve the diagnostic tasks, as the required level of knowledge and reasoning is within their development zone. This group may contain mathematically gifted children.

Mathematics aptitude can manifest itself at any time in any child whose mental capabilities are within the norm. Although the child does have to be helped in developing the mental attributes responsible for the development of aptitude (i.e., strong cognitive motivation and the ability of staying focused for longer periods of time, individuality and being driven by a sense of meaning, feeling satisfied by the pursuit of a goal, etc.).

Then there is the case of all the high-tier children with exceptional competences (most or all tasks solved exceptionally well). These children reason better than their peers, they know and can do more in regards to maths, and have no issues using the introductory tasks. They can be called mathematically gifted. The second segment focuses on a more in-depth analysis of their mental capabilities.

Short description of the second segment of the teacher diagnosis

This diagnosis encompasses all the children who know and can do a lot more than their peers. The goal is assessing whether they possess the mental attributes that are evident of a developing mathematics aptitude. The second segment consists of two diagnostic experiments titled: *Taking turns in creating and solving tasks* and *Intentionally misconstructed tasks*. Both were created as a research scenario.

The *Taking turns in creating and solving tasks* scenario contains a description of the conditions in which the teacher and the child are to take turns creating and solving instruction-based tasks, as well as teaching aids to be used in this experiment. The researcher and the child create and solve tasks in series: two for addition and for subtraction. The first task of the series for addition and subtraction is created by the researcher. It is very simple, aimed at five-year-olds. The aim is to convince the child that creating and solving tasks is easy. The child also has the possibility of creating a harder task for the adult. The child creates a task, the researcher solves the task. And vice versa. What's more, the scenario contains a description of the tasks presented to the child by the adult. When the series of tasks is nearing completion, the researcher intentionally makes a mistake while solving a task created by the child. The goal is whether the child will notice or not and how will they react.

During the process of creating and solving diagnostic tasks it is important to take notice of how the way the children taking part in the study manifest their mental attributes. The reactions and behaviour of the child are graded on a simple, three-level scale in the following scopes: a) stance towards maths activity, b) creativity in maths activity, c) a sense of meaning in counting and calculating.

The scenario of the *Intentionally misconstructed tasks* diagnostic experiment also specifies the conditions in which the diagnosis is to be carried out. The misconstructed tasks are then describe along with the presentation method. The researcher is to observe the reaction of the child. If the child is surprised, they are asked to fix the mistakes in the task and create a similarly misconstructed task for the researcher to solve. The child can either intentionally insert mistakes into the task, create a proper task (without any mistakes), or not create any task at all. This research procedure allows for assessing whether the child exhibits: a) a sense of meaning in maths activity, b) understanding the structure of instruction-based tasks at school, c) courage in regards to reacting to absurdities, e.g. when the child is presented with a misconstructed task by an adult. The second part of the diagnosis ends with tips regarding the way of assisting the development of mathematically gifted children at home, at kindergarten, and at school.

Interpretation and assessment of the levels of mathematics aptitude

The results of the first and second segments allow for precisely dividing the children into three groups. Exceptionally mathematically gifted children are those who show a lot of interest in creating and solving tasks while taking turns, are trying to impress with their skills, and want to create more tasks. This is a sign of well-developed creativity skills as well as happiness and satisfaction stemming from maths activity. They also exhibit an exceptional sense of meaning and the ability for the critical appraisal of maths activities. Not only do they notice how the mistakes were made in tasks, but they also try to correct them and create similar tasks. They also notice the mistakes made while solving tasks and know how to correct them. They are courageous in their thought formulation. What's more, they know and can do more when it comes to maths than their peers. There is no doubt about them being exceptionally mathematically gifted.

Mathematically gifted children are those who are better at counting and calculating than their peers and can enjoy maths activities. They are creative in their maths activity and create tasks in accordance with the given model. Their sense of meaning helps them notice mistakes when guided. They understand the structure of instruction-based tasks at school, but creating a misconstructed task is hard for them. They are reluctant to voice their concerns when facing a misconstructed task presented by an adult. This is why they are dependent on the adults in their maths activity and unsure of their mental capabilities. This can be remedied easily by encouraging them towards creative maths activity and creating a proper environment for them to gain knowledge and mathematics skills.

Average children are those who barely enjoy maths activity, even though they're proficient in counting and calculating. They are not creative enough yet to easily create tasks which require counting and calculating. They realise that something is amiss when facing misconstructed tasks, but they do not realise what exactly and do not try to correct them. They also do not notice the deliberate mistakes made when solving tasks. This may be caused by too much faith placed in adults, which results in thinking that the adults are always right. The reason why children choose to carry out their maths activity in this way is that the adults – parents and teachers – did not assist the child in developing their creativity. They also did not facilitate any sense of meaning from the children, as they thought children should not exhibit any kind of criticism towards maths activity.

From my pedagogical experience, when taking part in mental development classes and faced with an interesting maths education course, the children eventually get better. This is why these children should also be assisted in their development and maths education, and not only the exceptionally gifted.

Actions taken to improve the fate of mathematically gifted children

I based these actions on the work of great pedagogues [42] who left an important mark on education. After conceiving a concept, they tested it by performing a pedagogical experiment at kindergarten or at school while analysing the course and effects. Teachers were invited to such places – which can be called *islands of pedagogical happiness* – for them to understand what makes a pedagogical concept successful. If they liked what they saw, they created their own *islands of happiness*.

42. A good example is *The University of Chicago's Laboratory School* founded by J. Dewey in 1896.

This is why I keep organising meetings with headmasters, teachers, and parents [43]. I am offering to create *islands of pedagogical happiness.* They are being presented with the results of the study regarding mathematically gifted children presented in this work and a concept of maths education with emphasis on assisting the mental development of children. The concept contains my program, as well as the rules and methods of conducting lessons with children who are on their last year of kindergarten or first year of primary school [44]. The concept was created so that children can develop their minds and learn maths with accordance to their needs and developmental possibilities. Poorly-performing children perform as well as their peers, getting better and better month by month, while mathematically gifted children develop their abilities with no trouble.

At the request of parents and teachers I can provide my program and the methodology of conducting lessons with children. I also oversee the educational institutions performing the diagnoses. I oversee the diagnosis itself, I meet with the parents and teachers of the children. I have also organised postgraduate courses [45] to prepare teachers for the recognition of mathematics aptitude in children and planning their education with the development of their mental skills in mind.

How many *islands of happiness* have been created so far [46]? At the *Morska Kraina* kindergarten in Kołobrzeg, the children's mathematics aptitude development program has been in effect for four years so far, and Eureka, a school near Warsaw, has been teaching children in accordance with my concept for two years now. In Chorzów [47], 14 *islands* have been

43. I have found tremendous help in systematically organised conferences (proposed by my publishers of my works) and articles published in the press, e.g. *Każdy ma talent do matematyki* [*Everyone has a knack for maths.*] (Gazeta Wyborcza 11 September 2013), *Szkoła rzeźnią talentów* [*School, the talent slaughterhouse*] (Gazeta Wyborcza 29 May 2014).

44. What's more, I have experimentally tested the educational effectiveness of this concept by carrying out research as part of a) research project *Wspomaganie rozwoju umysłowego wraz z edukacją matematyczną dzieci w klasie zerowej i w pierwszym roku nauczania szkolnego* [*Assisting in mental development and the mathematics education of children in the last year of kindergarten and the first year of primary school*], no. H01F 083 30, financed by research funding in the years 2006-2009, b) the project *Rozpoznawanie i wspomaganie rozwoju uzdolnień do uczenia się matematyki u starszych przedszkolaków i małych uczniów* [*Recognising and encouraging the development of aptitude for learning mathematics in older kindergartners and primary school students*], no. R1700603, financed by research funding in the years 2007-2010. Reports from the studies are available at the Academy of Special Education in Warsaw.

45. They are conducted at the Academy of Special Education in Warsaw.

46. As of January 2014.

47. The City of Chorzów is very interested in changing the course of mathematics

organised in kindergartens, and it has been established that the children will continue their education in schools which will allow for the program of the assistance for the development of mathematically gifted children to be carried out.

These establishments publish the mathematics accomplishments of the children and their teachers on their website. Those interested in assisting in the development of maths aptitude in children can take a look, and, if they so desire, create their own *island of pedagogical happiness.* One of the most important arguments for the fact that this is a way of changing education for the better is that the headmasters and teachers of educational establishments in Zakrzew, Bytów, and Kutno are actively preparing for the experiment. Goleniów might be next, if I manage to convince the parents of kindergartners, the teachers, and the headmasters on this conference.

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education in kindergartens and schools. This is due to the fact that it is a good way of influencing the direction of the education of adolescents. If mathematics education does not seem scary to children, and tends to the development of their aptitude, and ensures successes in their education, they will show a desire in the future to choose majors that allow for an appealing profession. As it turns out, the umemployed in Chorzów are not ones with science degrees. It has been established that using my concept might help.

przedszkolnego i w pierwszym roku szkolnej edukacji. Cele i treści kształcenia, podstawy psychologiczne i pedagogiczne oraz wskazówki do prowadzenia zajęć z dziećmi w domu, w przedszkolu i w szkole. Warszawa: Wydawnictwo Edukacja Polska.

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Research results on mathematical talent, gender and motivation

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Abstract. In Germany, girls are decidedly underrepresented in programs that foster mathematical talent at primary school age. Thus, it is of interest to ascertain aspects improving their identification and support. Two studies were conducted to clarify the significance of motivational factors as determinants for the identification of talent by comparing girls and boys who were identified to be mathematically talented as well as girls and boys who were not. The first study focused on self-concepts, attributions and the number of interests, the second on attitudes and mathematics interest. The results indicate that the characteristics of all examined motivational factors were more advantageous with girls and boys who were identified to be talented as well as with boys who were not compared to girls who were not identified to be talented. Thus, disadvantageous motivational factors seem to be important aspects to explain the infrequent identification of girls' talent.

1. Introduction

In Germany, just like, e.g., in other western European countries, girls are in proportion decidedly underrepresented in programs that foster mathematical talent (Benölken, 2011). This phenomenon contradicts the consensus on the fact that both sexes do not differ in their cognitive abilities independently of certain domains (Endepohls-Ulpe, 2012). When it comes to primary school age, aspects like gender stereotyping of mathematical occupational fields rather cannot act as possible explanations,

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Key words and phrases: mathematical talent, mathematical giftedness, gender, motivation.

AMS (2000) Subject Classification: Primary 97B10, Secondary 97C20.

especially because there cannot be found any gender-specific differences in mathematical competencies at this age (Lindberg et al., 2010). In addition, studies have indicated a decline of such differences at subsequent ages for many years (Hyde et al., 2008). Thus, it is of interest to look for aspects improving the identification of girls' mathematical talent. With a holistic approach diagnostics should be organized as a process considering both cognitive and co-cognitive parameters as determinants in order to identify talent (cf. 2.). Among other factors, motivational constructs like "self-concept", "interest", "attributions" or "attitudes" play an important role in this context raising the question how they can be characterized with girls and boys who were identified to be mathematically talented ("imt") as well as with girls and boys who were not ("n-imt"). In this article, their significance as determinants in order to identify mathematical talent at primary school age will be examined by two questionnaire studies (Benölken 2014; 2015), whose aim is to look for boys' and girls' frequent characteristics by a comparison of the mentioned four groups. Based on literature reviews, hypotheses on possible characteristics will be deduced that correspond to the studies' questions. Afterwards, the design and the results of the studies will be reported and discussed.

2. Some preliminary notes about mathematical talent

According to Fuchs and Käpnick (2009), in this article "mathematical talent" is seen as an above-average potential regarding the criteria of Käpnick (1998), i.e. remembering mathematical facts, sensitivity and fantasy, structuring and transferring structures or reversing thoughts. This potential is characterized by individual determinants and a dynamic development depending on inter- and intrapersonal influences in interdependence with personality traits supporting the talent. Research on the regarded motivational factors mostly refers to "giftedness" as a "g-factor-concept" implying standardized diagnostics. Thus, their results cannot be transferred automatically to the reported view on "mathematical talent" regarding domain-specific criteria and implying long-term process diagnostics. Existing findings collectively show, however, the significance of the regarded motivational factors as determinants in order to identify girls' mathematical talent. Therefore, they are suited to provide a basis for the intended deduction of hypotheses.

3. Design and results of the first study

3.1. Theoretical frameworks and brief literature reviews

As to *self-concepts*, the conception applied in the study refers to Shavelson, Hubner & Stanton (1976): Self-concepts develop globally and domain-specifically containing both cognitive-evaluative and affective components. They can already be found at primary school age (Marsh, Craven & Debus, 1991). Studies show that as early as at this age, gifted and non gifted children differ in their global- and domain-specific self-concepts (Rost & Hanses, 2000). In contrast to global self-concepts (Rost & Hanses, 2000), there are findings about gender-specific differences in domain specific ones (Rustemeyer & Jubel, 1996). Boys, e.g., often show better self-concepts in mathematics (Pohlmann, 2005), girls in social or verbal skills (Valtin & Wagner, 2002). Among other things, disadvantageous mathematical self-concepts seem to be responsible for the fact that primary school girls do not tend do a strong preoccupation with mathematics (Dickhäuser & Stiensmeier-Pelster, 2003). Obviously, boys and girls who were identified to be gifted do not differ in their mathematical self-concepts (Wieczerkowski & Jansen, 1990).

In this article, *interest* is seen as a result of an interaction between a person and an object that – along with adjuvant conditions – causes to focus on a long-term preoccupation with it (Prenzel, Krapp & Schiefele, 1986; cf. 4.1.). A couple of studies show that even primary school children often have a lot of interests like sports, TV, computer games or reading (Pruisken, 2005). Furthermore, gender-specific differences can already be found at this age (Hoberg & Rost, 2000): horseback riding, animals or reading seem to be "typical" girls-, football, technics or computer "typical" boys-interests (Fölling-Albers, 1995). Boys even at primary school age more often show stronger mathematics interest; girls language or literature interest (Pruisken, 2005). Though gifted children show the same differences, they do not have any extraordinary interests, but gifted children generally seem to be more interested in both mathematics and languages or literature (Pruisken, 2005). In contrast to non-gifted girls, gifted girls have more interests which are supposed to be "typical" interests of boys, and they have a larger spectrum of interests than gifted boys (Kerr, 2000).

The construct of *attributions* refers to reasons that an individual provides to explain his or her achievements. They are basically divided into the dimensions of "locus of control" and "stability" (Weiner, 1986). Studies show that as early as at primary school age and irrespectively of certain domains, especially in mathematics, girls tend to attribute success external-unstably and failure internal-stably, i.e. disadvantageously. In contrast, boys tend to advantageous internal-stable attributions of success and external-unstable ones of failure (Rustemeyer & Jubel, 1996). Contemporary studies indicate that girls (even if they are gifted) more often tend to internal-unstable attributions of success, while boys still tend to internal-stable ones (Dickhäuser & Meyer, 2006; Tirri & Nokelainen, 2011). Gifted children generally attribute more advantageously than non-gifted children (Schütz, 2000).

3.2. Questions

The following hypotheses were deduced from the theoretical findings: (1) Imt girls and boys as well as n-imt boys show more advantageous mathematical self-concepts than n-imt girls. (2) Imt girls have a larger spectrum of interests than imt boys, n-imt girls and n-imt boys. (3) Imt girls and boys tend to internal attributions of mathematical success (3a.1). N-imt boys more often tend to internal attributions of mathematical success than n-imt girls (3a.2). Imt girls and boys tend to external attributions of mathematical failure (3b.1). N-imt boys more often tend to external attributions of mathematical failure than n-imt girls (3b.2).

3.3. Design

The study adds to Benölken (2011), especially to parts focusing on the significance of motivational factors as determinants in order to identify talent. Data of Benölken (2011) was used to enlarge the imt girls' sample because of their infrequent identification. Aiming at the composition of an instrument that is appropriate to primary school children and that can be completed in a short time, operationalizations of self-concept and attributions were extracted from a questionnaire that focuses on motivation beyond other domains and were put together with a short interest-questionnaire (in each case tested within pilot studies; Benölken, 2011).

3.4. Sample and procedure

The sample contains $N = 288$ children of the third and fourth grade (132 f, 156 m). The subsample of imt children is $n = 165$ (66 f, 99 m). Children who are assessed to be "mathematically talented" take part in a project that fosters mathematical talent at the University of Münster called "math for small pundits". They were chosen by long-term process-diagnostics that are a synthesis of standardized and non-standardized tools (cf. 2.; Benölken, 2014). From this group of probands 85 were questioned during the school year of 2012/2013 (35 f, 50 m). In addition to that, all children who completed the questionnaire of interests in the study of Benölken (2011) were included, i.e. $n = 80$ (31 f, 49 m), among them $n = 33$ probands (14 f, 19 m) whose data about self-concept and attributions could be clearly assigned. These probands were questioned during the school year of 2008/2009 using a non-anonymized questionnaire (as opposed to subsequent questioning). The sample contains $n = 123$ n-imt primary school-children (66 f, 57 m) from common classes questioned during the school year of 2012/2013. The n-imt group is obviously independent of the group of imt children. All procedures of questioning were consistent: The children were told how to fill in the questionnaire. They completed it on their own without any time limit (no one took more than 15 minutes and no one refused to fill in the questionnaire).

3.5. Method

Apart from declaring sex, the questionnaire was anonymized. In order to measure self-concepts by both a cognitive-evaluative and an affective aspect, the following instruction was given: "Mark with a cross a statement that is proper to you: [1] I am very good at math. [2] I particularly enjoy solving difficult math-tasks". To evaluate the items in each case a four-step Likert-scale was offered ("that's not correct", "that's almost not correct", "that's almost correct", "that's correct"; instead, the children could choose "I don't know"). To collect data about the number of interests, a schedule according to the above mentioned research results was composed intending to offer a large spectrum of interests (cf. Benölken, 2014). Beyond that, further interests could be added into open lines. The instruction was: "Mark with a cross all interests that you have. In the open lines you can also note interests that are not mentioned". Attributions of success were operationalized by the instruction: "Imagine: You solved a difficult math-problem. Why did you succeed? Because... [1] you worked really hard, [2] it was random, [3] you're very good at math, [4] the task was simple". Attributions of failure were analogically operationalized (cf. Benölken, 2014). Just one answer was allowed to be chosen to get the strongest trend. Instead "another reason" could be added for both success and failure in an open line. These answers could be assigned to the dimensions of Weiner (1986) afterwards in all cases.

3.6. Evaluation

Statements about self-concept-items were translated into numbers from 1 ("that's not correct") to 4 ("that's correct"). The coefficient of correlation as defined by Pearson between these items is .588 ($p < .001$) and the internal consistency is acceptable or even good (Cronbachs $\alpha = .73$). The items have been combined to one scale with mean values. The chosen interest-items have been transformed into one variable containing their sum. Data about self-concepts and interests have been evaluated by an analysis of variance with two factors ("talent" and "sex") to find significant differences between the four groups. In addition to that, η^2-values have been calculated to see the importance of both the factors and their interaction by their effect size. As to the evaluation of attribution-data, cross-tabs have been built containing talent, sex and the dimensions of Weiner (1986). They also include standardized residua to point out significant differences: Values ≤ -1.96 or ≥ 1.96 indicate an ascertainable divergence from expected frequency in each cross-tabs-cell regarding to a level of significance of $\alpha = .05$ (Eid, Gollwitzer & Schmitt, 2011). The significance of possible differences was tested using the exact Fisher-test: According to Weiner's dimensions, attribution was operationalized by a nominal scale consisting of four values. Data of imt and n-imt children were evaluated independently. For remarks about requirements of all statistical procedures see Benölken (2014).

3.7. Results

	self-concepts		interests	
	boys	girls	boys	girls
imt children	3.58 (.44) $n = 69$	3.60 (.42) $n = 49$	8.68 (3.43) $n = 99$	12.44 (3.82) $n = 66$
n-imt children	3.33 (.59) $n = 57$	2.58 (.87) $n = 66$	7.44 (2.47) $n = 57$	11.11 (3.21) $n = 66$

Table 1. Averages (standard deviations) of self-concept-statements and of interests' sum.

Table 1 shows averages and standard deviations of both self-concept-statements and the total sum of interests. As to *self-concepts*, there are significant main effects on talent ($F(1, 237) = 63.39$, $p < .001$, $\eta^2 = .211$) and sex ($F(1, 237) = 21.16$, $p < .001$, $\eta^2 = .082$) as well as a significant effect of interaction ($F(1, 237) = 23.80$, $p < .001$, $\eta^2 = .091$). Thus, there is a main effect on sex which cannot be interpreted because the averages of imt boys and girls are nearly identical. As indicated by η^2-values, talent (strong effect) plays a bigger part to explain variance than interaction between talent and sex (medium effect). Therefore, imt children have more advantageous self-concepts in comparison with n-imt children, but n-imt boys merely differ a little. This fact confirms hypothesis 1.

Looking at the *number of interests*, there are significant main effects on talent ($F(1, 284) = 10.50$, $p = .001$, $\eta^2 = .036$) and sex ($F(1, 284) = 86.77$, $p < .001$, $\eta^2 = .234$), but there is no effect of interaction ($F(1, 284) = .01$, $p = .915$, $\eta^2 = .000$). Sex (strong effect) plays a bigger part to explain variance than talent (small effect). Hypothesis 2 is confirmed for imt girls, but n-imt girls have more interests compared to the two groups of boys on average, too.

		internal-unstable		internal-stable		external-unstable		external-stable	
		s	f	s	f	s	f	s	f
imt boys	number	21	15	42	1	2	16	4	37
	residua	−.5	−1.0	.6	−.2	.8	.2	−.8	.6
imt girls	number	19	18	23	1	0	10	6	20
	residua	.6	1.2	−.7	.2	−.9	−.2	.9	−.8
n-imt boys	number	25	15	31	0	1	9	0	33
	residua	−8	−4	2.8	−2.5	−1.2	1.0	−2.4	1.0
n-imt girls	number	38	21	10	13	6	5	12	27
	residua	.7	.4	−2.6	2.3	1.2	−.9	2.2	−.9

Table 2. Cross-tabs about descriptions of attributions of mathematical success (s) and failure (f)

Table 2 shows attribution-data. As to *attributions of success*, hypothesis 3a.1 cannot be confirmed or rebutted since the Fisher-test is not significant ($= 4.044$, $p = .243$). In contrast, hypothesis 3a.2 was confirmed by a significant result ($= 30.137$, $p < .001$). Compared to n-imt boys, n-imt girls more infrequently tend to internal-stable (-2.6 to 2.8), but more often to external-stable (2.2 to -2.4) attributions as shown by the standardized residua. With regard to *attributions of failure*, hypothesis 3b.1 cannot be confirmed or rebutted because the Fisher-test is not significant ($= 3.656$, $p = .282$), but hypothesis 3b.2 was confirmed ($= 19.882$, $p < .001$). In comparison with n-imt boys, n-imt girls more often tend to internal-stable attributions as shown by the standardized residua (2.3 to -2.5).

4. Design and results of the second study

4.1. Theoretical frameworks and brief literature reviews

As already mentioned in 3.1., the applied conception of *interest* refers to Prenzel, Krapp, and Schiefele (1986). More detailed, this relation is characterized by (1) value-related, (2) affective, and (3) cognitive aspects. Additionally, in accordance with current approaches on a multidimensional structure of interests a distinction by subject-, context- and topic-related interest was regarded (for a survey see Krapp, 2010). The first two dimensions were summarized in the term of "mathematics interest in the classroom" because it cannot be expected that primary school children differ between activities and contexts applied in classrooms (see also Hellmich, 2006). The third dimension is referred to by the term "mathematics interest beyond the classroom". In addition to the findings that have been reported in 3.1., boys more often show stronger mathematics interest both in and beyond the classroom, though children at primary school age do not differ between these aspects in large part (Hellmich, 2006). However, current studies do not focus on gender-, giftedness- or talent-specific aspects in this context. Furthermore, there are only a very few studies with a focus on ability-related mathematics interest. Their findings indicate, that lower achievement students' mathematics interest exceeds that one of higher achievers (Frenzel et al., 2010), but these studies do not focus on gifted or talented students. Finally, an often reported phenomenon is a decline of mathematics interest over the years of adolescence (Fredricks & Eccles, 2002), which is of little importance at primary school age.

The construct of *attitudes* focuses on an evaluation of objects which an individual imagines or perceives in his or her environment. Attitudes can be explicitly and consciously accessed or they can emerge implicitly and spontaneously influencing an individual's behavior in both cases (Bohner, 2003). The conception applied in the study refers to the classical operationalization consisting of (1) cognitive, (2) affective and value-related, as well as (3) behavior-related components (Aronson, Wilson & Akert, 2004). Generally, studies indicate that male students more often show advantageous characteristics of mathematics attitudes than female students (Hyde et al., 1990). As to the cognitive aspect, studies primarily focus on individuals' assessments of usefulness and difficulty of mathematics. While there seem to be no gender – or talent-specific differences between imt and n-imt children regarding usefulness (Benölken, 2011), some studies indicate that mathematically gifted boys and girls as well as non gifted boys ascribe mathematics a lower level of difficulty compared to non gifted girls (Wieczerkowski & Jansen, 1990). Finally, there are findings about gender stereotypes: The older girls are the more they ascribe mathematics to males (Newton & Newton, 1998), which seems to be less important at primary school age since such differences mostly appear from an age of ten. Concerning the affective aspect, results on gender- or giftedness- respectively talent-specific differences of individuals' intrinsic values (like enjoying mathematical task solving) seem to play the most important role: Similar to characteristics of the assessment of mathematics' difficulty, some studies indicate that mathematically gifted boys and girls as well as mathematically non gifted boys show a higher intrinsic value doing mathematics compared to mathematically non gifted girls (Wieczerkowski & Jansen, 1990), while other studies report that boys ascribe mathematics a higher intrinsic value than girls in general (Bos et al., 2012). In regard to the behavior-related aspect, boys seem to engage in mathematics beyond mathematical school lessons more often than girls (Schiepe-Tiska & Schmidtner, 2013).

4.2. Questions

The following hypotheses were deduced from the theoretical findings: (1a) Imt girls and boys as well as n-imt boys show a stronger mathematics interest in the classroom than n-imt girls. (1b) Imt girls and boys as well as n-imt boys show a stronger mathematics interest beyond the classroom than n-imt girls. (2) Imt girls and boys as well as n-imt boys show more advantageous mathematics attitudes than n-imt girls.

4.3. Design

The study adds to previous research on the significance of motivational factors as determinants for the identification of mathematical talent using questionnaires that are appropriate to primary school children and that can be completed in a short time (cf. 3.3.). Operationalizations of mathematics interest in and beyond the classroom as well as of attitudes were tested within pilot studies.

4.4. Sample and procedure

The sample contains N= 162 children of the third and fourth grade (71 f, 91 m). The subsample of imt children is $n = 83$ (32 f, 51 m). Children who are assessed to be "imt" take part in the project "math for small pundits" (cf. 3.4.). The sample contains n=79 n-imt primary school children (39 f, 40 m) from common classes. The probands were questioned during the school year of 2014/2015. All procedures of questioning were consistent: The children were told how to fill in the questionnaire. In this context, possible differences between mathematics interest in and beyond the classroom were emphasized (cf. Benölken, 2015). The children completed the questionnaire on their own without any time limit (no one took more than ten minutes and no one refused to fill in the questionnaire).

4.5. Method

Apart from declaring sex, the questionnaire was anonymized. In order to measure mathematics interest in the classroom by a value-related, an affective and a cognitive aspect, the following instruction was given: "This is about mathematics in the classroom. Mark with a cross a statement that is proper to you: (1) Mathematics in the classroom is really important to me. (2) I always look forward to mathematics in the classroom. (3) I am interested in mathematics in the classroom". An analog instruction was composed to collect data about mathematics interest beyond the classroom (cf. Benölken, 2015). To measure attitudes by cognitive, affective and behavior-related aspects, the following instruction was given: "Mark with a cross a statement that is proper to you: (1) Mathematical tasks are sometimes too difficult. (2) I enjoy doing mathematics. (3) I engage in mathematics beyond mathematical school lessons". To evaluate the items in each case a four-step Likert-scale was offered ("that's not correct", "that's almost not correct", "that's almost correct", "that's correct"; instead, the children could choose "I don't know"; cf. 3.5.).

4.6. Evaluation

Statements about all items except the cognitive attitudes one were translated into numbers from 1 ("that's not correct") to 4 ("that's correct"). As to the cognitive attitude item, the assignment was turned around: For instance, "that's not correct" was translated into 4 and "that's correct" into 1, because statements that focus on a low level of mathematical tasks' difficulty seem to reflect advantageous characteristics of attitudes. Regarding the mathematics-interest-in-the-classroom-scale, the coefficient of correlation as defined by Pearson between the included items moves in a range from .366 to .475 (with p<.01 in each case) and the internal consistency is only just acceptable (Cronbachs $\alpha = .680$). As to the mathematics-interest-beyond-the-classroom-scale the coefficient of correlation as defined by Pearson between the included items is in a range from .378 to .576 (with $p < .01$ in each case) and the internal consistency is between acceptable and good (Cronbachs $\alpha = .731$). Finally, the coefficient of correlation as defined by Pearson between the included attitudes items moves in a range from .334 to .617 (with $p < .01$ in each case) and the internal consistency is between acceptable and good, too (Cronbachs $\alpha = .710$). The evaluation was conducted analogically to the evaluation of self-concepts and sum of interests within the first study by an analysis of variance with the two factors "talent" and "sex" (cf. 3.6.; the requirements of the used statistical procedure are discussed by Benölken, 2015).

4.7. Results

	mathematics interest in the classroom		mathematics interest beyond the classroom	
	boys	girls	boys	girls
imt children	3.07 (.83) $n = 51$	2.89 (.51) $n = 32$	3.39 (.70) $n = 51$	3.37 (.48) $n = 32$
n-imt children	3.30 (.76) $n = 40$	2.65 (.72) $n = 39$	3.33 (.79) $n = 40$	2.74 (.55) $n = 38$

Table 3. Averages (standard deviations) of mathematics interest-statements

Table 3 shows data of mathematics interest-statements. As to *mathematics interest in the classroom*, there is no significant main effect on talent ($F(1, 158) < .001$, $p = .990$, $\eta^2 < .001$), but there can be found a significant main effect on sex ($F(1, 158) = 12.795$, $p < .001$, $\eta^2 = .075$) as well as a significant effect of interaction ($F(1, 158) = 4.139$, $p = .044$, $\eta^2 = .026$). As indicated by η^2-values, sex (medium effect) plays a bigger part to explain variance than the interaction (medium effect). Thus, the boys' groups, especially the n-imt boys, show a stronger mathematics interest in the classroom compared to the girls' groups, but as indicated by the significant effect of interaction, imt girls are more similar to the boys' groups than to the n-imt girls who show a lower mathematics interest in the classroom on average compared to all other groups. Therefore, the statistical evaluation confirms hypothesis 1a in principle.

As to *mathematics interest beyond the classroom*, there are significant main effects on talent ($F(1, 157) = 10.579$, $p = .001$, $\eta^2 = 063$) and sex ($F(1, 157) = 8.435$, $p = .004$, $\eta^2 = .051$) just as there is a significant effect of interaction ($F(1, 157) = 7.579$, $p = .007$, $\eta^2 = .046$). Talent (medium effect) and sex (medium effect) play a similar role to explain variance. Thus, imt children and n-imt boys show similar characteristics of mathematics interest beyond the classroom which is stronger compared to n-imt girls, and hypothesis 1b is confirmed. In addition, a descriptive data analysis of all groups' mean values indicates that only imt children seem to differ between mathematics interest in and beyond the classroom.

	boys	girls
imt children	3.32 (.58) $n = 51$	3.11 (.60) $n = 32$
n-imt children	3.03 (.78) $n = 40$	2.23 (.76) $n = 39$

Table 4. Averages (standard deviations) of mathematics attitudes-statements

Table 4 shows data of *mathematics attitudes*-statements. There are significant main effects on both talent ($F(1, 158) = 29.023$, $p < .001$, $\eta^2 = 155$) and sex ($F(1, 158) = 21.550$, $p < .001$, $\eta^2 = 120$). Finally, there is a significant effect of interaction ($F(1, 158) = 7.597$, $p = .007$, $\eta^2 = .046$). Talent (strong effect) and sex (medium effect) play a similar role to explain variance, even though the talent effect is stronger. Thus, attitudes of imt children are more advantageous compared to n-imt children, but n-imt boys merely differ a little from the imt children. The statistical evaluation confirms hypothesis 2.

5. Discussion

In this article the significance of self-concepts, interests' number, attributions, mathematics interest – by a distinction between in and beyond the classroom – and attitudes as determinants in order to identify mathematical talent at primary school age was investigated by a comparison of frequent characteristics with boys and girls who were identified to be mathematically talented (imt) as well as with boys and girls who were not (n-imt). Based on literature reviews, hypotheses on the mentioned characteristics were deduced: It has to be expected that (1) imt children and n-imt boys show more advantageous characteristics of the regarded motivational factors than n-imt girls, and (2) imt girls have more interests than all other groups. The hypotheses were investigated by two questionnaire studies. The statistical results confirm the assumptions in principle (though, e.g., girls in general seem to have more interests than boys). Thus, the results are very similar to existing findings focusing on "giftedness" (e.g., as to the number of interests similar to Kerr, 2000). With regard to mathematics interest in and beyond the classroom, only imt children seem to differ between these dimensions showing stronger interest beyond the classroom, while n-imt children took similar stances in both cases (which could explain the results of Hellmich, 2006).

As to the significance of the regarded motivational factors as determinants in order to identify talent, the results indicate that more advantageous characteristics can be found independently of the identification of talent more often with boys, while imt girls are very similar to these groups. This might cause more efficient diagnostics of boys' talents, because they might tend to a strong preoccupation with mathematics or teachers might perceive their potentials primarily. By contrast, disadvantageous characteristics might cause that children do not develop a stronger preoccupation with mathematics and, e.g., turn to different interests. This might also apply to children who have a high potential that might be more difficult to identify. Though the findings are not suitable to predict how the regarded motivational factors can be characterized with talented but not identified girls, the results imply the thesis that disadvantageous characteristics are important aspects effecting a more infrequent identification of talent with girls. In addition, motivational factors have to be seen in a strong interdependence with, e.g., influences of socialization or gender-specific preferences in solving tasks (Benölken, 2011).

As to limitations of the studies and subsequent research, the underrepresentation of girls in the imt samples has to be discussed: Because of the rare identification of mathematical talent with girls, it takes a long time to compose suitable subsamples. Though the size of all subsamples is sufficient in principle, subsequent studies should enlarge all subsamples and ensure a balance. The diagnostics procedures of talent identification that are used to compose the imt subsample are established for many years. Thus, "imt" children most probably are rightly assessed in that way. In addition, there might be motivational effects caused by their participation in "math for small pundits" that cannot be found with children who have high potentials, but who are not taking part in such a program. Finally, the subsamples of n-imt children are nothing more than an insufficient image of population. Thus, the samples' representativeness has to be seen as limited. The questionnaires were adequate to the aims of the studies in principle. They are suited for a pragmatic use in classrooms because their design is appropriate to children, and they can be completed in a short time. However, the evaluation of the regarded motivational factors depends on very simple measurements. The external validity of the findings cannot be judged because tools that evidentially regard criteria of quality were not applied (in favor of the appropriateness to young children) and because the imt sample is very specifically composed. In sum, the study has obvious limitations, and it rather has an explorative character. Subsequent studies might focus on a deeper clarification using established tools.

As to exemplary practical consequences, first, any gender stereotyping of mathematics has to be avoided. Second, the development of advantageous motivational factors seems to be important in order to support girls' potentials to emerge. In this context, e.g., task-fields that are composed especially to foster girls – without stereotyping – might be useful (Benölken, 2013). The distinction between mathematics interest in and beyond the classroom that was observed with imt children indicates the need of a challenging education, e.g., by using enrichment tasks in common classes (Fuchs & Käpnick, 2009).

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Prediction and self-evaluation as a part of the process of solving non-standard mathematical task

Abstract. The experience obtained from solving the tasks of the Mathematical Kangaroo in the category of primary school pupils is a rich source of inspiration for the educational practice. In the article, the process of solving Kangaroo tasks is described in the context of the more general issue of meta-cognition. We present the results of a research aimed at finding the level of prediction and self-evaluation of performance by primary school pupils during the process of solving a set of non-standard Mathematical Kangaroo tasks.

1. Introduction

The theory and practice of education confirm that the analysis of pupils' solutions of mathematical problems can be a useful instrument of a more accurate, more meaningful assessment of pupils based on the thorough knowledge of the pupils' individuality (Hejný, Kuřina 2001, Kalinowska 2012, Dabrowski 2013). The solution of non-standard tasks thus represents a possible source of data about the educational reality as well as its reflection.

Many authors such as Schoenfeld (1992), Kopka (2007), Novotná (2000), Fulier and Šedivý (2001) consider these non-standard tasks more interesting, even though they are usually more difficult than the standard routine tasks. They emphasize that non-standard tasks lead to discovering, inventing, and finding new ways to solve a problem and thus develop the cognitive abilities of pupils, and that the analysis of the pupils' solu-

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Key words and phrases: non-standard tasks, solution, prediction, self-evaluation, elementary school pupils..

AMS (2000) Subject Classification: 97B10.

tions of tasks from the Kangaroo competition can be a useful instrument of a more complex assessment of the pupils' individuality.

2. Theoretical background

In our research, we combine the solution of mathematical problems with the more general issues of metacognition, i.e. "the ability to reflect one's own processes of thinking and ways of improving one's thinking" (Sternberg, 2002, p. 215), as we believe that this relationship can be scientifically rooted (Schoenfeld, 1992). We focus on the self-evaluation and prediction of the pupils' performance in the sense of the "reflection of an action" and label them as "off-line" metacognition (Desoete et al., 2001).

A number of cognitive processes that are necessary for successful solution take place during the process of solving the tasks. When encountering difficulties during the process of solution, metacognitive processes are used - Fisher (1997) speaks of metacognitive pupils who think about their thinking and focus on the task, know what to do when they get stuck, and are successful in using their strategies. Opinions vary both on the levels of metacognition found among primary school pupils and on how to develop the metacognitive processes of pupils.

One can rarely see research on the metacognition and self-regulation of children so young, and their occurrence is not very elaborate, although some authors – Perry, Drummond (2002), and Larkin (2000) - state that pupils of primary-school age already achieve a certain level of metacognition, meaning that they are able to plan, monitor, and evaluate their own learning.

In our research, we were inspired by the research of Zgarbová (2011). We focused on an attempt to investigate prediction and self-evaluation as part of the solution of non-standard mathematical problems. The international Math Kangaroo competition is coordinated by the Center Association Kangourou sans frontières based in Paris. It is intended for pupils aged 8-18. Every year, more than 3 million solvers register. In the Czech Republic, there are annually approximately $300,000$ participants, and in the Ecolier category, nearly $70,000$ pupils in $4th$ and $5th$ grade of elementary school take part. The competition does not have any rounds – on the same day, participants from more than 50 countries around the world solve the same tasks in their respective age categories. Tasks are classified into three levels of difficulty. The author of this paper is the guarantor of the Ecolier category in the Czech Republic and the Czech version of the competition tasks for that category. For more details about the organization of the competition, its history, and the competitive tasks in individual years, see the international or Czech website of the competition.

3. Objective, research method and tools

The aim of our research was to identify the level of "off-line" metacognition (i.e., the prediction rate and level of self-evaluation) of the pupils in 4th and 5th grade and to determine whether it depends on their success in solving non-standard mathematical problems. We assumed that students successful in solving problems achieve a significantly higher prediction rate and level of self-evaluation than the unsuccessful students. When formulating research questions and hypotheses, we operationalized the variables of the research:

a) The pupils' performance as the success rate of solving non-standard mathematical problems: the total number (sum) of points from the solution of the competition test consisting of 10 tasks. The correct answer was evaluated at 2 points, partially correct at 1 point, incorrect or missing answer at 0 points. Each respondent could amass 20 points. Based on their success rate, the solvers were divided into successful (20-10 points) and unsuccessful (9-0 points),
b) The prediction rate of pupils related to the solution of non-standard mathematical problems, i.e. the comparison of the perceived ability and the actual performance (max. 20 points),
c) The level of the pupils' self-evaluation, i.e. the comparison of the subsequent perception of success in solving problems and the actual performance (max. 20 points).

As the basic research technique, we used a didactic test consisting of 10 tasks, which included questions aimed at identifying the prediction rates of pupils and their levels of self-evaluation. The Ecolier tasks were subsequently adapted into open test items. Each task has exactly one correct solution. We selected tasks of lower difficulty levels: 8 tasks rated in the competition at 3 points and 2 tasks at 4 points. The factor that unifies the diversity of the content of the tasks (tasks requiring arithmetic calculations, the concept of fractions as parts of a whole, a task requiring space orientation skills) and the method of presentation (word or text supplemented by image, contextually defined - as a result of the mathematization of the real situation) was their non-standard character. The test was identical for pupils in both 4th and 5th grade of elementary school (aged 10 – 11 years).

Instructions for pupils:

This test contains 10 non-standard mathematical tasks from the previous years of the Mathematical Kangaroo competition. Its aim is to find out whether you can solve these tasks. Please complete the test in following way:

1. *In the test, you will find mathematical problems. Read all the tasks from* 1 *to* 10*, but do not try to solve them yet.*
2. *Prediction: Try to estimate whether you are able to solve each task. Tick your prediction for every task. Move from task* 1 *to task* 10*.*
3. *Solution: Now, try to solve the tasks. Under the wording of each task, write your solution. You can take a blank piece of paper for your notes.*
4. *Self-evaluation: Finally, tick the answer in the table indicating how you think you solved each task. Proceed again from task* 1 *to task* 10*.*

Here are two examples of the tasks:

a) *Soňa threw a die four times and she obtained a total of* 23 *spots. How many times did she get* 6 *spots?*

b) *Below, six coins form a triangle. You have to move some coins to place them in a circle, as you can see in the second picture. How many coins must be moved at the very least?*

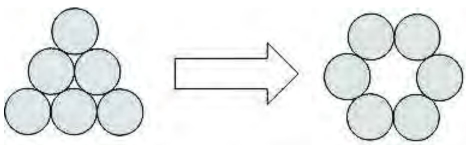


Figure 1. Task.

To each task in the test, one question has been connected with prediction and another with self-evaluation, which made a total of 10 questions examining the degree of prediction and self-evaluation each.

When evaluating the degree of prediction and self-evaluation, we did not take into account the sum of points ticked by the pupils on the scale (i.e., their subjectively perceived value), but the real measure of their prediction and self-evaluation. This means that we compared the prediction with their actual performance in solving the test tasks (separately for each task). For example, if a pupil had estimated that he would solve the task correctly and indeed he did, then the pupil was awarded 2 points. If a pupil considered his correct solution only as probable and solved the task correctly, then the pupil was awarded 1 point. When the pupil was sure that the task was correctly solved but did not solve it, there was no point awarded. The relationship between the prediction and the actual success of the pupil in the solution of the task (score prediction rate) is described in Table 1.

Prediction	**Performance**	
	Correct solution	Incorrect or no solution
I will definitely solve the task correctly	2	0
I will likely solve the task correctly	1	0
I will likely not solve the task correctly	0	1
I will definitely not solve the task correctly	0	2

Table 1. Relationship between prediction and actual performance of pupil.

Analogically, we proceeded in terms of the relationship between the pupil's self-evaluation, made immediately after solving the problem, and the real performance - the level of self- evaluation scores are given in Table 2:

Self-evaluation	**Performance**	
	Correct solution	Incorrect or no solution
I definitely solved the task correctly	2	0
I likely solved the task correctly	1	0
I likely did not solve the task correctly	0	1
I definitely did not solve the task correctly	0	2

Table 2. Relationship between self-evaluation and actual performance of pupil.

The survey was conducted on a set of 204 pupils of 16 primary schools in four regions of the Czech Republic (random choice) in November 2014. The research focused on two grades ($4th$ and $5th$) of each participating school. The sample consisted of 54 pupils aged 9 years (26.9%), 109 pupils aged 10 years (53.4%), and 41 pupils aged 11 years (20.1%).

The obtained data have been processed via a quantitative methodological approach. The graphic depiction (tables and bar charts) and position characteristics calculation were applied when referring to descriptive nature of data.

4. Findings and discussion

In our research, we found a low level of success of solving non-standard tasks - an average of 8.39 percentage points (i.e., only 4 successfully solved task out of 10). Only 6 solvers (2.94%) correctly solved all 10 problems, and 9 solvers (4.41%) were not able to correctly solve any problem. One of the causes of this, which we consider alarming, we see largely in the fact that the solutions required the verbal understanding of the formulated open test tasks. The correct solution is not based on a calculation routine, but rather requires a clear insight into the situation in the task and the application of mathematical abilities. It is also a known fact that many teachers, for various reasons, prefer using standard algorithmic problems (Novotná, 2000, Rendl et al., 2013). The results of the research seem to confirm the view reflecting the previous experience during an educational practice in primary school – the solution of non-standard problems is not a usual and frequent activity in mathematics classrooms. As it is apparent from some studies (Swoboda, 2014), this finding applies generally, and not only for a typical situation of Czech education.

Descriptive Statistics

	N	Mean	Minimum	Maximum
Successful	81	13.80	10	20
Unsuccessful	123	4.92	0	8
Total	204	8.39	0	20

Table 3. Success rate of non-standard task solution.

When we look at the sub-processes of metacognition, i.e. the prediction rate and level of self-evaluation on the sample of all respondents, we find that the values are aligned and relatively low. The overall level of prediction was 8.04 points from a total of 20 points and the overall level of self-evaluation amounted to 8.85 points from a total of 20 points.

In the group of successful pupils (10 or more points gained), the average performance was 13.80 points, prediction 9.16 points, and self-evaluation 10.38 points. In the group of unsuccessful pupils, significantly lower average values were achieved: performance 4.92 points, prediction 7.23 points, and self-evaluation 7.89 points. Successful pupils achieved significantly higher levels of prediction rate as well as levels of self-evaluation than unsuccessful pupils. The differences are statistically significant. We conclude that students successful in mathematics, who have higher levels of mathematical ability and prove this by solving non-standard tasks, are also able to predict and objectively evaluate their performance.

Descriptives

	N	Mean	Std. Deviation	Std. Error	Minimum	Maximum
Successful	81	9.16	3.219	.569	4	18
Unsuccessful	123	7.23	2.779	.170	2	16
Total	204	8.04	3.169	.114	2	18

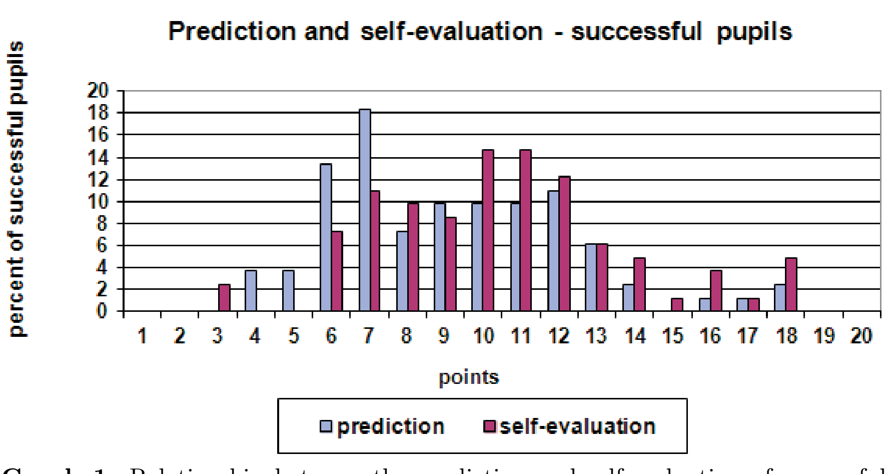
Table 4. Overall level of prediction.

Descriptives

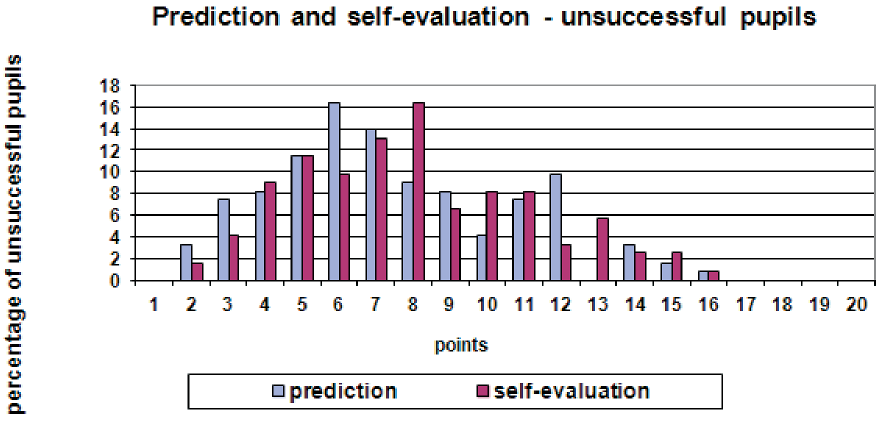
	N	Mean	Std. Deviation	Std. Error	Minimum	Maximum
Successful	81	10.38	3.621	.500	3	18
Unsuccessful	123	7.89	2.828	.166	2	16
Total	204	8.85	3.415	.122	2	18

Table 5. Overall level of self-evaluation.

The relationship between the prediction and self-evaluation of pupils, and their success in solving problems is evident from graphs 1 and 2:



Graph 1. Relationship between the prediction and self-evaluation of successful pupils.



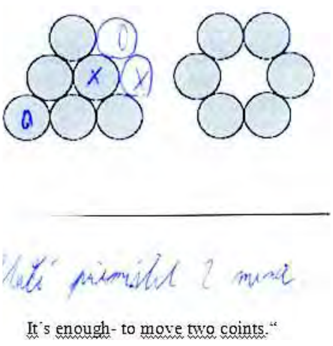
Graph 2. Relationship between the prediction and self-evaluation of unsuccessful pupils.

Our findings largely correspond to the conclusions of the aforementioned research of Zgarbová (2011).

We can indirectly conclude that the school does not focus on how to teach students to think and to learn. Planning, monitoring and self-evaluation can have a big impact on the pupil's success at school. However, pupils have little experience with this metacognitive approach (Zgarbová, 2011, p.128).

The structure of the research also allows to observe the prediction, performance, and self-evaluation of particular pupils, both generally as well as in detail, regarding a particular task.

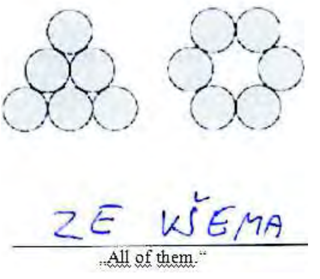
The pupils' solution of the two tasks:



Picture 1. Vojta's solution.

In the task, Vojta got 2 points for prediction and 2 points for self-evaluation. In this case, it means that in the prediction, Vojta chose the possibility I will definitely solve the task correctly, and he solved the task correctly. His self-evaluation was I definitely solved the task correctly, and indeed he did. This is the reason he got 2 points for both (prediction and self-evaluation).

In the whole test, Vojta got 10 points for prediction, 11 points for self-evaluation and 14 points for his performance.



Picture 2. Jana's solution.

In this task, Jana got 0 points for prediction and 0 points for self-evaluation. In this case, it means that when predicting, Jana chose the possibility I will likely solve the task correctly, but she did not. Her self-evaluation was I likely solved the task correctly, but she was not correct. In whole test, Jana got 7 points for prediction, 10 points for self-evaluation, and 2 points for performance.

The Mathematical Kangaroo tasks are suitable for the employment of prediction as well as self-evaluation. Pupils start the test with 24 points. Each incorrect answer leads to the loss of 1 point, and correct answers add the respective amount of points based on the difficulty (3, 4, 5 points). There is a time limit for the solution of the test. Pupils apply prediction after reading the task wording when evaluating whether to start to solve the task or for some reason, e.g. because it is too difficult or too time consuming, to skip to another task. Self-evaluation also has its place in the competition. When a pupil solves a task, a decision must be made of whether it is the correct solution, because only then it makes sense to state it in the list of results. Obviously, the uncertainty regarding the correctness is not taken into consideration here (I likely solved the task correctly, I likely did not solve the task correctly), but in any case, pupils still have the choice of not stating the answer at all.

5. Conclusion

In the article, we attempted to use the tasks from the previous years of the Mathematical Kangaroo competition to suggest one possible approach to the prediction and self-evaluation of the performance of pupils in primary schools.

We did not analyze the impact of other potential variables that could influence the success of solving problems, the prediction rate, or the level of self-evaluation: the personal characteristics of the respondents – gender, mathematics grade, or the popularity of the subject, neither did we consider the types of tasks, their difficulty, theme, nor the type of task instruction. Some of these factors, in relation to the successful solution of the problem, have been the subject of more widely designed research of Kubátová (2005).

The features of our probe or sample size of respondents does not allow for unambiguous categorical judgments. However, the findings can be, in our opinion, definitely considered an impulse and an inspiration not only for primary school teachers.

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Career construction in the mathematics classroom: using an integrated, qualitative + quantitative approach to enhance learners' sense of self

Abstract. Career Construction Counselling and Self-Construction Counselling aim to help learners script their career-life stories. This approach is suitable for exploring personal meanings and for helping learners deal with many of the problems involving meaning. The aim of the paper is to demonstrate the implementation of an integrated, qualitative+quantitative approach in career counselling in combination with a parallel approach in mathematics to elicit and harness the learners' reflection and facilitate reflexivity, enhance their sense of self and, ultimately, enable them to participate more actively in their career and self-construction and in the writing of their emerging career-life stories. Ten practical principles for facilitating self-construction (previously referred to as optimization of potential/self-actualization in mathematics classrooms) are discussed. These principles can promote not only better achievement in mathematics but, more importantly, enable learners to construct themselves adequately and design successful lives that will enable them to make useful social contributions. It is concluded that drawing on an integrated, qualitative + quantitative approach in mathematics classrooms holds much potential to enable the learners in a variety of contexts to improve their mathematical performance and, more importantly, help them to make and execute informed decisions about their career-life journeys.

Key words and phrases: Career counselling, the implementation of an integrated, qualitative + quantitative approach in career counselling and mathematics, achievement in mathematics, mathematical performance, career construction, sense of self, career-life stories.

AMS (2000) Subject Classification: Primary 91E10, Secondary 91E99.

In psychology fresh winds are blowing . . . picking up and juxtaposing separate ideas to produce novel combinations
(Tyler, 1978, p. 1)

Introduction

The world of work is changing more rapidly today than ever before – many careers are disappearing, and new ones are taking their place virtually daily. Predicting career journeys is consequently becoming increasingly difficult. In the United States of America (USA), for instance, people are changing jobs more or less once every four and a half years and often have 10 to 15 jobs before they reach the age of 38 (Vecchio, 2013). Finding suitable employment is also more problematical today than previously. Whether we are responding adequately to fundamental changes in the field of ICT (information communication technology; fifth wave of information – also known as the digital revolution (Gurri, 2013)), the 4$^{\text{th}}$ Industrial Revolution, the global economy, and the world of work is a question that needs to be asked (Maree, 2015).

We as mathematics teachers need to rethink our theory and practice continually in order to help learners deal with the challenges stemming from the sweeping changes in the world of work and to survive in a world characterized by uncertainty in the workplace. Teaching, learning, psychology, and career counselling theories are not static as every forty to fifty years theoretical paradigms tend to reach the limit of their relevance and usefulness (Savickas, 2011). We therefore need to consider whether psychology and mathematics have responded appropriately and timeously to the changes in the world in general, in society (including the social organization of work), and in the economy. A brief review of the teaching and learning theories that have guided teaching and learning in mathematics over the past 100 years reveals that theorists and practitioners have repeatedly shifted backwards and forwards between theoretical models such as the 'traditional model', cognitivism, information processing, and a number of variations broadly based on a constructivist approach (e.g. (social) constructionism).

Much can be learned from recent developments in career counselling whose orientation has shifted from overly 'positivist' to 'positive' during the past few decades. This major paradigm shift can be attributed largely to the tireless efforts of Savickas (2011) who merged the person-environment fit (differential) paradigm with the developmental and narrative/ psychodynamic/ storied paradigms resulting in a true meta-theory, referred to as career construction counselling. Savickas also devised an associated assessment and intervention strategy known as the career construction interview. In doing so, he changed the course of career counselling and demonstrated the value of an integrated, quantitative + qualitative approach as opposed to a quantitative approach in relative isolation. Career construction counselling and self-construction counselling are premised on the view that it is better to help learners script their career-life stories rather than testing them and telling them what career to choose.

This is important for the psychology of mathematics as it is generally accepted that mathematics is the pre-eminent gateway subject for selection to tertiary studies. Ernest (2015) contends that "mathematics is a dynamic, growing modern city and not a skyscraper" and that learners should be equipped in mathematics classrooms with the skills needed to deal with the changes outlined above. This paper therefore argues that the implementation of an integrated, qualitative+quantitative approach in career counselling, in combination with a parallel approach in mathematics to elicit and harness learners' reflection and reflexivity, can enhance their sense of self and, ultimately, enable them to participate more actively in career and self-construction and in the writing of their emerging career-life stories.

Research Questions

The main research questions in my study were:

a. Do we, as mathematics teachers adequately reflect on, learn from, and build on our past endeavours?, and
b. How can mathematics teachers best convert the challenges brought about by global changes into opportunities to solve the problems of a rapidly changing world?

Aim of the Paper

This paper:

a. considers the meaning of an integrated, qualitative+quantitative approach to learning facilitation in mathematics classrooms and lecture halls, and
b. elaborates on the importance of using career construction in mathematics classrooms to enhance learners' and students' sense of self.

Research Design

In my study, I followed a meta-theoretical approach by drawing on the results of previous studies. Since no systematic cataloguing of research in the psychology of mathematics had been done, the research was based on a qualitative analysis of scientific texts. Firstly, a literature review was undertaken of various research projects conducted by the author of this paper and his colleagues during the past few years (e.g., Steyn & Maree, 2002; Maree, Molepo, Owen & Ehlers, 2005; Maree & Eiselen, 2007; Van der Walt & Maree, 2007; Maree, Van der Walt & Ellis, 2010; Maree, Fletcher & Sommerville, 2011; Maree, Lombard, Fletcher & Sommerville (ongoing)). The articles and reports that flowed from the research were considered narratives, and the approach was seen as an alternative research method. Secondly, discourses between the researcher and various participants were analysed and interpreted. In terms of this approach (discourse analysis), the problem-saturated narratives were also analysed and placed in context. I endeavoured to show the connection between various studies, and I also proposed a number of theoretical and intervention approaches.

Meta-Theoretical Framework

I believe Boboc (2011) is correct in saying that it is essential to reconcile the *pétit récit* ('small story/narrative') and the *grand récit* ('big story') by focusing on the bigger picture (Boboc, 2011). In other words, we as mathematics teachers should respond to challenges by basing our practice on a conceptual framework that can inform and shape our actions. I believe we should base our practice on the following sextet of T's, which will help mathematics teachers "control the controllables" and "resolve the resolvables" in mathematics classrooms.

a. <u>T</u>eacher (training)
b. <u>T</u>eachee (learner)
c. <u>T</u>eaching and learning
d. Paren<u>T</u>
e. <u>T</u>extbook
f. <u>T</u>eaching time.

Our practice should be guided by the following simple principles:

a. Teachers must teach; learners must learn (obvious as this may sound).
b. 'Best practice' in mathematics classrooms can best be facilitated by moving from a 'positivist' approach towards a 'positive' approach to teaching and learning in mathematics.
c. Teaching should be considered a value-driven activity, and reciprocal respect, punctuality, and self-discipline should be displayed at all times.

Some Thoughts on 'Transgression'

I share the view of the organisers of this ground-breaking conference on the importance of enabling learners to 'transgress' (in other words, transcend generally accepted boundaries) when they attempt to master mathematics and use mathematics to create better lives for themselves and for others. Sarrazy (2015) contends that today's learners will have to re-invent the world of tomorrow and go beyond what they learn today if they are to survive in the 21st century. He adds that learners should be allowed the experience of transgression as it will be difficult for them to excel if they have never had the opportunity to experience how it feels to move beyond obvious boundaries in the mathematics classroom.

From Theory to Practice and from Practice to Theory

I concur with Savickas (2009) that theory follows practice and vice versa and with Schoenfeld (2015) when he says that mathematics teachers can regard themselves as successful only if they can effectively cover the following in their classrooms.

a. Mathematical content;
b. Cognitive demand;
c. Access to mathematical content;
d. Agency, authority, and identity;
e. Uses of assessment period.

However, I believe that a sixth and a seventh dimension should be added to the list, namely promotion of emotional stability and volitional robustness in mathematics learners.

In the next subsection of this paper, I discuss ten practical principles for facilitating self-construction (Savickas, 2011) (previously referred to as optimization of potential/self-actualization (Maslow, 1983)) in mathematics classrooms. These principles can promote not only better achievement in mathematics but, more importantly, enable learners to construct themselves adequately and design successful lives that will enable them to make useful social contributions.

Principle 1: Move from intention to action (Savickas, 2009)

Mathematics teachers should move

a. from being reactive to pro-active,
b. towards facilitating 'best practice' in one-on-one settings AND in group contexts,
c. towards optimizing communication, creatively combining

i. electronic/mobile (SMS, cell phone, WhatsApp) communication,
ii. social communication media,
iii. paper-based communication,
iv. workshop-style communication, and
v. one-on-one communication.

Principle 2: Facilitate reflection

Reflection should occur during every mathematics class, including

a. reflection-in-action (during teaching),
b. reflection-on-action (after teaching), and
c. reflection-for-action (for teaching) (Farrell, 2004; Killion & Todnew, 1991).

Principle 3: Facilitate reflexivity

The aim of reflection (looking back on learners' thoughts and actions) is to facilitate reflexivity on their part (planning for the future).

Principle 4: Learn from and build on the past

Teaching and learning in mathematics has switched between various philosophies and the practical application of theoretical models, including discovery learning as well as learner-, teacher-, subject-, and variations of problem-centred learning. Also, education authorities often include modules in learning programmes only to remove them at a later stage when shortcomings become evident in their implementation. A case in point was the overhasty, politically driven introduction of an outcomes-based approach to teaching and learning in mathematics in South Africa. Theorists and practitioners should learn from their mistakes and take steps to prevent the recurrence of the same mistakes.

Principle 5: Help learners understand the importance of dealing with unfair relationships in education and society

I agree with Ernest's view (2015) that mathematics personifies openness, equality, fairness, and justice, which is sufficient motivation for us to teach the ethics (and philosophy) of mathematics in every classroom and during every class.

Principle 6: Help learners recognize opportunities (not only solve problems)

In the 21st century, learners are offered employment on the basis of their ability to generate ideas and make the ordinary extraordinary. It is therefore our task to teach them to think innovatively and creatively. Wittmann (2015) elaborates on this notion by arguing that 'productive practice' should characterize all mathematics classes. In other words, it is essential to integrate the practicing of skills in mathematics classrooms with the exploration, examination, and investigation of mathematical problems.

Principle 7: Instil a positive attitude and hope for the future in learners

Research suggests that learner achievement is enhanced if learning takes place in an environment of positive expectations engendered by lecturers, parents, and role models (McIlveen & Midgley, 2015; Neault, 2013).

Principle 8: Shift the focus from the individual learner to the group

This recommended action is consistent with Schoenfeld's (2015) view that the time has come to shift from focusing on individual learners' thinking only and, rather, consider the dynamics of the classroom as a whole.

Principle 9: Facilitate emotional-social intelligence (ESI) skills (Goleman, 1996)

It should be noted first of all that ESI can be acquired and developed (improved). Learners can be taught to retain or discontinue using particular adaptive coping strategies depending on situational demand. ESI, including self-motivation, the ability to persevere in the face of failure, the ability to postpone immediate needs satisfaction in order to satisfy long-term needs, and the ability to hope and to prevent sorrow, concern, and anxiety from interfering with thought processes. The importance of acquiring acceptable ESI is borne out by findings that confirm that ESI is a far better predictor of achievement and success than mere IQ or aptitude. Pieronkiewicz (2015), too, argues that the importance and power of affective transgression integrated with cognitive transgression cannot be denied.

Principle 10: Facilitate narration/discourse/dialogue in mathematics classrooms

In addition to assessing learners 'quantitatively' and devising programmes to address challenges (shortcomings or problems) uncovered by these assessments, it is essential also to focus on qualitative ways of uncovering challenges and dealing with them. This can be achieved by using questionnaires, facilitating immediate feedback from and reflection by learners, facilitating meta-cognitive thinking and action (executive behaviour) on their part, and by using reflective journals in mathematics classrooms.

Conclusion

Mathematics teachers should exploit change to advance positive teaching and learning in mathematics – the ultimate aim of theory and praxis in the field. It seems clear that an integrated, qualitative+quantitative approach in mathematics classrooms can help learners in a variety of contexts improve their mathematical performance and, more importantly, help them make and execute informed decisions about their career-life journeys. In other words, help them go beyond generally accepted *perceived* boundaries in mathematics classes.

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Applying cognitive load theory in mathematics education

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Abstract. To characterize the quality of mathematics education in a country is not an easy task. The high quality of mathematics education in Hungary has often been assumed, due to Hungary's tradition of producing world-famous mathematicians. For example, outside of Hungary, one often hears, "Hungarian mathematics problem-solving teaching is world-famous thanks to George Pólya", and contributions of other excellent Hungarian mathematicians, such as J. Neumann, P. Halmos, E. Szemerédi and L. Lovász, are often noted. While it is true that Hungary has produced many great mathematicians, the author's premise is that this does not establish the general quality of Hungarian mathematics teaching, or imply that it cannot be improved. On the contrary, the author believes that mathematics education in Hungary can be improved. To evaluate this premise, I analyse the results of Hungarian students who took the international PISA 2012 mathematics test, as well as two national tests of grade 12 students and the mathematics proficiency of new college entrants. Based on the results, this article asks: How can we prepare these students for success in their mathematics studies and their future jobs more effectively? Based on the human cognitive architecture and cognitive load theory, my hypothesis is that "opening" mathematics problems and providing more guidance would amount to two steps toward this important goal. After some theoretical considerations, I analyse some of the students' performance during the experimental trials of this idea, and suggest some steps toward successfully implementing it to more

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effectively teach the fundamentals of mathematics problem-solving to students in Hungary. Finally, I summarise my conclusions.

1. Characteristics of hungarian mathematics teaching

The characteristics of current Hungarian mathematics teaching may be the result, in part, of the views of three great Hungarian mathematicians and educators. G. Pólya, the "father of heuristics", made Hungarian mathematics teaching strongly problem-oriented. Tamás Varga led the "Complex Mathematics" reform movement in Hungary in the 1970s and 1980s. Finally, Z. Dienes stated the following, a motto of a mathematics education conference in Eger, 2015: "In my opinion, it is an attainable goal to build creative learning environments on all levels of mathematics learning. If a child successfully developed a concept based on his(her) own experiences, then he(she) created something that did not exist earlier for him(her), and it got ingrained in his(her) psychological personality, just like his(her) essential food materials are incorporated into his(her) body. This piece of knowledge will have as much value for him(her) as painting a picture, or writing a good story" (Dienes, 1999).

These great mathematicians and educators all supported discovery/problem-based mathematics education, and made significant contributions both to the teaching of mathematics and mathematics research in Hungary and the world. However, the effectiveness of the Hungarian mathematics education system today is lagging. This may be one of the reasons why J. Mason noted: "The much-vaunted Hungarian mathematics teaching [system] has not spread significantly into the mainstream" (Mason, 2013).

The effectiveness of the Hungarian mathematics teaching philosophy and system can be debated, but objective indicators of the system's effectiveness are indicated by the Hungarian students' performance on recent international and national achievement exams.

On the international *PISA 2012* Mathematics test, out of 65 country participants, Hungary ranked in 39^{th} place, with an average of 477 points (OECD average was 494 points). Only 9.3% of Hungarian students attained the top rating of 6 and 28.1% of Hungarian students were rated at lower levels 2 or 1. On the *PISA 2012* Creative Problem Solving test, out of 44 country participants, Hungary ranked 33^{rd} and 35% of Hungarian students were rated at lower levels 2 or 1. Hungary has always participated in PISA tests right from the beginning, usually achieving OECD average results, but these latest results show a decline.

On the two national tests of upper level mathematics proficiency, the Hungarian students' performance has been similarly below average.

The first national test is the mathematics *central maturity exam*, which is given to students at the end of their secondary school year 12. This test has two levels, a "higher" and a "middle" level, representing different levels of mathematical expertise. The higher-level test has both a written and an oral component, and includes differential and integral calculus as well as complex problems, including many modeling problems. The middle-level test includes only basic mathematical tasks, algorithms, or procedures. Every year, approximately 95 000 Hungarian students take this exam. Of this total, only approximately 3 500 (or 3.6%) students feel confident enough in their mathematics skills to take the higher-level test. However, universities training future engineers, information technologists, mathematicians, mathematics teachers, and architects only have enough space to accommodate approximately 20–25% of each graduating class, and a high level of mathematical knowledge, as would be tested by the higher level central maturity exam, is a requirement for these majors.

The other test is the *first year test* that all students must take at the beginning of their university or college-level studies. This test contains problems based only on middle-level central maturity exam requirements. Currently, student results on this test require most students to participate in a remedial "adjustment" course in order to continue with their studies, in which they review basic secondary mathematical concepts, algorithms, procedures, etc. Two elite Hungarian universities (Technical University Budapest and Eötvös Lóránd University Budapest) recently discovered similar indications of Hungarian students not achieving the desired mathematical competence. These universities found that most students do not understand key mathematical relationships and ideas; they have weak analyzing, modeling, and imaginative abilities; their work is hard to follow, and their knowledge is superfluous. It does not need to be emphasized how important these factors are in effective mathematical problem solving (Csákány, 2011).

To summarize: The above indicators show that the majority of Hungarian students lack basic desired mathematical proficiency. Therefore, the Hungarian mathematics education system is not working very effectively and needs to be improved. While the fostering of the most talented students' mathematical skills and understanding is not an issue (these students' performance remains high), the system is not teaching the vast majority of students the mathematical knowledge, skills, and problem solving abilities that they need. Laurinda Brown, who visited Hungarian secondary schools many times, may have hinted at the problem years ago when she stated: "You in Hungary are teaching mathematics, we in England children!" (Brown, 1990). More recently, the Mathematical Transgressions 2015 Cracow conference suggested new approaches to improving the teaching of mathematics, with lessons from different disciplines to improve the effectiveness of mathematical teaching at primary and middle school levels. Many of the discussed approaches seemed to possibly be applicable in Hungary. In any case, it is clear that we need to change this situation.

What can mathematics educators do to effectively teach not just the top 5–10% of the most gifted students, but to teach all students basic math skills, and give at least the top quartile of mathematically-talented students the more advanced math skills that they need to succeed in their studies and the increasingly technical career opportunities of the future?

As a basis for discussing this question, I will review, first, human cognitive architecture and memory systems, and second, Cognitive Load Theory and its implications for problem-based mathematics teaching. Finally, I will analyze the results of some practical experiments that were conducted in Hungarian schools considering cognitive loads and potentially decreasing them in order to improve the students' learning of mathematics, and the implications of these experiments to improving mathematics education in Hungary.

2. The human cognitive architecture and memory systems

Most neuroscientists accept Baddeley's model of memory structures: perceptual (sensory) memory, working memory, and long-term memory (Baddeley, 2009). Similarly, cognitive architecture has been extensively researched, and its implications for learning are generally acknowledged: "Any instructional procedure that ignores the structures that constitute cognitive architecture is not likely to be effective. Minimally guided instruction appears to proceed with no reference to the characteristics of working memory, long-term memory, or the intricate relations between them. The result... require(s) learners to engage in cognitive activities that are rightly unlikely to result in effective learning..." (Kirschner, 2006).

Therefore, the consideration of cognitive architecture and memory structures is critical for effective teaching. For educators, working memory and long-term memory are the most important ones, as they are the core locations where human cognition takes place. They can be described as follows:

Working memory is where conscious human information processing occurs (e.g., comprehension, understanding, critical thinking, problem solving, etc.). It is the "workbench" of our brain; the active problem-solving space. Working memory has four components: the phonological loop (to hold and rehearse verbal information); the visual-spatial sketchpad (to hold visual and spatial information); the episodic buffer (which connects the verbal and visual-spatial information) and the "central executive" (a supervisory attention system that monitors, controls, and directs information processing between these areas, with the help of information taken from long-term memory). To summarise, working memory constructs plans, uses transformation strategies, analogies, and metaphors, brings thoughts together, and abstracts and externalizes mental representations. Working memory, therefore, is extremely important in problem solving, as the students need a clear mental representation of the task at hand (understanding the problem), and, while seeking a strategy (solution method), need to remain aware of the conditions and the goal, monitor progress, inhibit wrong, unsuccessful ideas, and control their results. All of these processes occur in the working memory. The working memory, however, has a very limited capacity. It can hold 7 ± 2 new information units, maybe four plus minus one. Additionally, working memory can process only two or three information units at a time. Finally, working memory can only hold information, without rehearsal, for about 18–20 seconds. Therefore, the limitations of working memory are very important in problem solving, as maintaining relevant information and inhibiting irrelevant information are critical (Baddeley, 2009; Clark, 2006).

Long-term memory is the storehouse of knowledge. It holds information in "schemas", mental structures that organize and structure knowledge. Schemas are created in working memory, and then integrated into existing schemas in long-term memory. Individuals then retrieve schemas (which hold knowledge) from long-term memory into their working memory as needed, in order to understand and process situations and problems. Unlike working memory, long-term memory does not have maximum capacity or time limit for holding information.

The connection between long-term and working memory is very important in learning. When schemas are built, they take information units from working memory, and free working memory resources. Schemas can also become "automated", and when individuals use automated schemas, there are no working memory capacity demands.

3. Cognitive load theory

Cognitive load can be defined as the load imposed on working memory by presenting information. It is based on the following assumptions

(Chipperfield, 2006):

- Short-term memory (working memory) capacity is limited to 7 informational units.
- Long-term memory (where all information and knowledge is stored) capacity is unlimited.
- Knowledge stored in long-term memory is held in "schemas", or schemata.
- Schemas, no matter how large or complex, are treated as a single informational unit in working memory.
- Schemas can become automated, imposing no load on working memory.

Because working memory is limited, the cognitive load placed on working memory must be limited in order for optimal learning to occur.

3.1. Types of cognitive loads

Intrinsic cognitive load (also sometimes called "essential processing") refers to the cognitive load imposed by information which must be processed simultaneously. For example, when solving word problems, this information would consist of reading the problem, concluding what the problem asks, and solving the problem. Intrinsic cognitive load is embedded in a problem; teachers cannot influence it.

Extraneous cognitive load refers to the cognitive load imposed by the manner in which information is presented. This may include unnecessary superfluous information (such as background music), holding mental representations of facts or figures, or separating related information (such as geometric figures and related written statements). Extraneous cognitive load can make information harder to understand, and is not embedded in a problem; therefore, teachers can influence it.

Germane cognitive load (also sometimes called "generative processing") is the cognitive load placed on working memory by schema formation, integration, and automation. Germane cognitive load explains the observed differences in the students' performance reflecting their relative experience, ability level, and content knowledge.

In summary, total cognitive load = intrinsic load + extraneous load + germane load. When planning teaching, teachers must take both potential total cognitive loads imposed by problem-solving as well as the instruction method into consideration, as too much cognitive load will impede learning.

3.2. Measuring cognitive load

There are three general methods of measuring cognitive load: subjective, physiological, and task-performance based.

The subjective method of measuring cognitive load is very realisable in the classroom. It is based on the assumption that students can assess the mental effort they are expending. An often-used technique of subjectively measuring cognitive load is the one-dimensional ninth grade symmetrical category scale developed by Pass (1992). In this technique, the students rate their perceived mental effort after completing a problem on a nine-point rating scale (ranging from "very, very low mental effort" to "very, very high mental effort").

The physiological method of measuring cognitive load includes measuring heart rate or eye activity while students are solving problems.

Task-based and performance-based methods of measuring cognitive load consist of measuring primary task performance (actual task performance) and secondary task performance (based on a secondary task, performed concurrently with the primary task), using a relevant scale.

3.3. Instructional designs to reduce cognitive load

As cognitive load on working memory must be limited for optimal learning, instructional methods to reduce and control cognitive load must be included in education. Instructional methods and principles relevant to reducing cognitive load include the following:

Worked examples. "Research has provided overwhelming evidence that, for everyone but experts, partial guidance during instruction is significantly less effective than full guidance" (Clark, 2012). The use of "worked examples" is a technique in which the solution to a problem is explained to the students and modelled by the teacher. This allows the students to concentrate on the essential problem states and possible related moves. It also facilitates the students' integration of solution schema into their long-term memory.

Open problems ("goal free" problems). For some problems, the distance between the start phase and the goal is very high. With such problems, it is desirable to ask the students to find all the relevant data they can during the process of solving the problems. In our experiments, the "opening" of closed problems goes in this direction.

Completion problems. These are a special form of worked examples. In such problems, there are gaps in the presented solution, and the students are asked to fill them.

Split-attention effect. For different representations (visual and textual) of the same concept, procedure, or strategy, the students need to create mental representations. This requires them to "split" their attention. If the representations of related information are very far from each other, it may be difficult for them to integrate these representations into a single mental representation that would allow them to best learn. Therefore, in teaching, it is desirable to present visual and verbal information in a way that facilitates the students' integration of this information into a single mental representation. (This explains why students will learn better from multimedia lessons when words are spoken, rather than printed, beside a picture.)

Modality effect. This refers to managing the essential mental processing of different forms of information (and also explains why people learn better from a multimedia lesson when words are spoken, rather than printed, beside a picture).

Redundancy effect. This occurs when multiple sources of the same information are self-contained, and can be understood alone, as the other form may disturb the understanding.

3.4. The implications of cognitive load theory for mathematics problem-solving teaching

The goal of mathematics education is to enable students to be successful mathematics problem-solvers. We shall analyse the two different basic positions on how to accomplish this goal: discovery/unguided instruction versus guided instruction.

In Hungary and perhaps elsewhere, mathematics educators have been strongly influenced by the ideas of Pólya, Dienes, and Varga that problem-based, unguided "discovery" learning is the best method of learning and teaching mathematics. For mathematics experts, and perhaps the top 5% of naturally mathematically-gifted students, the problem-based system of learning is clearly very effective. However, it is less effective for the remainder of students. This is indicated by recent test scores of the majority of Hungarian students taught by this method on the aforementioned international and national tests of mathematical competence. Additionally, it is explained by the commonly acknowledged principles of memory structure, cognitive architecture, and cognitive load theory. Therefore, an exclusive commitment to a teaching system based on closed problems and discovery learning can impede our goal as educators to teach all students, as well as ensure that the next 15-20% tier of students who have the aptitude and interest to pursue advanced studies in mathematics and/or professional technical or teaching careers that require advanced mathematics training, will be well prepared to do so.

After witnessing the difficulties of many of my own students in learning mathematics using the problem-based discovery method, I began my own research regarding human cognitive architecture and the findings of comparative studies of discovery/unguided versus guided instruction, and my research led me to change my own previous commitment to only make use of the discovery/unguided model of instruction.

The international literature in didactics of mathematics and educational psychology present a variety of views on this subject. Researchers who have conducted scientifically-controlled randomized studies in this area and whose findings I reviewed include John Sweller, Richard E. Mayer, John Hattie, Richard E. Clark, Paul A. Kirschner, and Gregory Yates.

J. Hattie summarized extensive research about learning and teaching in a study examining the experiences of 1 000 schools and 3 000 teachers in Australia and New Zealand, as well as 50 000 research articles and 800 meta-analyses. This research provided information on the learning of 240 000 000 students. One of the investigated issues concerned our topic of unguided versus guided instruction; or, teacher as "activator" versus teacher as "facilitator". The average teacher as activator effect size was 0.61, while the average teacher as facilitator effect size was 0.19 (with effect size = [average on post-test – average on pre-test] : standard deviation). The effect sizes found for teacher as activator included: feedback 0.75; teacher clarity 0.74; direct instruction 0.59; providing worked examples 0.57. The effect sizes found for teacher as facilitator included: inquiry-based teaching 0.31; problem-based learning 0.15; discovery methods in math instruction 0.11 (Hattie, 2014). Therefore, teacher as facilitator (i.e. guided learning) was found to be preferable to teacher as activator (i.e. unguided learning) as a method of teaching.

Research regarding cognitive load theory, as studied by Sweller, Clark, and Kirschner, confirm the implications of Hattie's results: "Research has provided overwhelming evidence that, for everyone but experts, partial guidance during instruction is significantly less effective than full guidance" (Clark, 2012). "The examples Pólya used to demonstrate his problem-solving strategies are fascinating and his influence can probably be sourced, at least in part, to those examples. Nevertheless, in over half a century, no systematic body of evidence demonstrating the effectiveness of any general problem-solving strategies has emerged. It is possible to teach learners to use general strategies such as those suggested by Pólya (Schoenfeld, 1985), but that is insufficient. There is no body of research based on randomised, controlled experiments indicating that such teaching leads to better problem solving" (Clark et al, 2012).

Regarding Bruner: "Recommending partial or minimal guidance for novices was understandable back in the early 1960s, when acclaimed psychologist Jerome proposed discovery learning as an instructional tool. At that time, researchers knew little about working memory, long-term memory, and how they interact. We now are in a quite different environment; we know much more about the structures, functions, and characteristics of working memory and long-term memory, the relations between them, and their consequences for learning, problem solving, and critical thinking. We also have a good deal more experimental evidence as to what constitutes effective instruction: controlled experiments almost uniformly indicate that when dealing with novel information, learners should be explicitly shown all relevant information, including what to do and how to do it" (Clark et al, 2012).

A number of reviews of empirical studies have established a solid research-based case against the use of unguided learning. While an extensive review of those studies is outside the scope of this article, Mayer (2004) has recently reviewed evidence from studies conducted from 1950 to the late 1980s comparing 'pure discovery learning', defined as unguided, problem based instruction with guided forms of instruction. He suggests that in each decade since the mid-1950s, when empirical studies provided solid evidence that the then-popular unguided approach was not working, a similar approach popped up under a different name with the cycle then repeating itself. Each new set of advocates for unguided approaches seemed either unaware or uninterested in the previous evidence which stated that unguided approaches had not been validated. This pattern produced discovery learning, which gave way to experiential learning, which, in turn, gave way to problem-based and inquiry learning, which is now giving way to constructivist learning. Mayer concluded that "The debate about discovery has been replayed many times in education but each time, the evidence has favoured a guided approach to learning" (Kirschner et al, 2006).

3.5. Regarding the effectiveness of problem solving learning and teaching

"The superiority of chess masters comes not from having acquired clever, sophisticated, general problem-solving strategies but rather from having stored innumerable configurations and the best moves associated with each in long-term memory. De Groot's results have been replicated in a variety of educationally-relevant fields, including mathematics (Sweller & Cooper, 1985). They tell us that long-term memory, a critical component of human cognitive architecture, is not used to store random, isolated facts, but rather to store huge complexes of closely integrated information that results in the problem-solving skill. That skill is knowledge domain-specific, not general. An experienced problem solver in any domain has constructed and stored a huge number of schemas in long-term memory that allows problems in that domain to be categorized according to their solution moves. In short, the research suggests that we can teach aspiring mathematicians to be effective problem solvers only by helping them memorize a large store of domain-specific schemas. The mathematical problem-solving skill is acquired through a large number of specific mathematical problem-solving strategies relevant to particular problems. There are no separate, general problem-solving strategies that can be learned. How do people solve problems that they have not previously encountered? Most employ a version of means-ends analysis in which the differences between the current problem-state and goal-state are identified and the problem-solving operators are found in order to reduce those differences. There is no evidence that this strategy is teachable or learnable because we use it automatically" (Sweller, 2011).

The problem with problem-based learning: If a student does not have the problem-corresponding schema, the student must search their working memory to find a relevant solution process. The means-ends analysis technique is a strategy to control such a problem-solving search. Given the difference between the current state and the goal state, an action is chosen which will reduce that difference. The action is performed on the current state to produce a new state, and the process is recursively applied to this new state and the goal state. This search in means-ends analysis causes a heavy burden for working memory, and if nothing happens in long-term memory, there will be no learning. The alternative is using worked examples. It enables students to concentrate on problem states and possible solution steps, and to transfer solution schema into long-term memory for later retrieval.

The class teaching method: The "class teaching" method is dominant in Hungarian mathematics education, and so is the tradition of using the so-called "problem-oriented" style. However, the effectiveness of this method and style are not proven. "In real classrooms, several problems occur when different kinds of minimally guided instruction are used. First, often only the brightest and most well-prepared students may disengage. Second, others may copy whatever the brightest students are doing – either way, they are not actually discovering anything. Third, some students believe they have discovered the correct information or solution, but they are mistaken and so they learn a misconception that can interfere with later learning and problem solving. Even after being shown the right answer, a student is likely to recall his or her discovery – not the correction. Fourth, even in the unlikely event that a problem or project is devised and all students succeed in completing minimally guided instruction is much less efficient than explicit guidance. What can be taught directly in a 25-minute demonstration and discussion, followed by 15 minutes of independent practice with corrective feedback by a teacher, may take several class periods to learn via minimally guided projects and/or problem solving" (Clark, 2012).

About constructivism: "The most recent version of instruction with minimal guidance comes from constructivism, which appears to have been derived from observation that knowledge is constructed by learners and so (a.) they need to have opportunity to construct by being presented with goals and minimal information, and (b.) learning is idiosyncratic and so a common constructional format or strategies are ineffective. The constructivist description of learning is accurate, but the instructional consequences suggested by constructivists do not necessary follow. Most learners of all ages know how to construct knowledge when given adequate information and there is no evidence that presenting them with partial information enhances their ability to construct a representation more than giving them full information. Actually, quite the reverse seems most often to be true. Learners must construct a mental representation or schema irrespective of whether they are given complete or partial information. Complete information will result in a more accurate representation that is also more easily acquired. Constructivism is based therefore on an observation that, descriptively accurate, does not lead to a prescriptive instructional design theory or to effective pedagogical techniques" (Kirschner et al., 2006).

4. Experiments with Hungarian students

What can be done to incorporate the lessons of the above research into Hungarian mathematics education? For one, we can mention one difficulty that is related to the use of closed problems: many students cannot even attempt solving closed problems without assistance, because these problems require top-down deductive reasoning. For these students, opening a problem gives them the opportunity to take individual steps toward reaching the solution and begin to learn (for example, by investigating concrete cases, which utilizes bottom-up, inductive reasoning). As a result of my research, I decided to conduct my own independent research regarding student learning and the responses to closed and open problems, as well as the effects of unguided and guided instruction in Hungary. My student samples included students of varying mathematics abilities and aptitudes of middle and secondary school and university level. I also had the opportunity to include in my experiments not just Hungarian students, but some Polish and Finnish students as well. Many of the students who participated in my research came from small cities and towns, and experienced large cultural differences with the students who attend elite schools in Budapest. However, they want to study at higher levels, and will be competing with such students. Therefore, the responses of all of the students who participated in this research were of great interest to me. The following is an example of a closed problem taken from a typical Hungarian mathematics exercise collection. I used questions taken from these actual examples in my research.

4.1. Closed Problem 1

The sum of three integers is 2014. Is it possible that their product is 111111?

Grade 5:

The 6 participating students tried to solve the initial, closed-form problem. Not one student could start. The students then were given a modified version of the problem (The sum of three integers is 10; can their product be 27? Hint: try to find 3 numbers with their sum being 10, and build the product of the summands. Look for more possibilities! Do you notice something? Give an argument!), with the following results. For 3 students, the modified problem was very unusual. “What shall I do?” asked V. For these 3 students, it was necessary for me to give 3 concrete numbers with the sum 10. Only then could they build other examples. The other 3 students could find more solutions for the sum and built the products, but they needed additional help: “What kind of numbers are the products?” Only M noticed the right patterns; namely, that there were two possibilities for the members of the sum: either all three numbers are even, or two numbers are odd, and one is even. So their product will always be an even number, and will never be an odd number.

Grade 6:

The 2 participating students tried to solve the problem. They could not solve the initial, closed-form problem. The students were then given the modified version of the problem, with the following results. For the sum 10, they found more sums, and they built the products. However, they needed help, and asked questions: “What is common between the products? Why? What kind of numbers are the members of the products?" After they received the answers to these questions, they gave the right arguments.

Grade 7:

The 1 participating student tried to solve the problem. He could not solve the initial closed-form problem. The student was then given the modified version of the problem, with the following results. With 10, the student could find more cases and correctly noticed the possible kinds of summands: 3 even, or 2 odd and 1 even. He also gave the right argument for the products. Interestingly, the students initially only used positive integers. It was necessary to ask them to choose negative integers, too. It was only after instructing them to choose negative integers that they saw that the same argument worked here, too.

Grade 10:

The 1 participating student tried to solve the closed problem. He divided 111111 into prime factors. He received $111111 = 3{\cdot}7{\cdot}11{\cdot}13{\cdot}37$. He also explained, "We should divide the five factors into 3 groups and build the sums and check whether it is 2014. Trying to choose all possible triples from the numbers above we see that it is impossible to get 2014 for their sum. But it is not nice work". What was not very nice for this student was that not only the prime factor triples, but all the possible factor triples should be checked, and that leads to combinatorics questions.

When I provided the student with some guidance and suggested, "Try with smaller numbers", the student very quickly found the impossibility of his original version, and made the right arguments.

Grade 11:

The 1 participating student was very bright, and tried to solve the problem. He correctly explained, "Of course, it is not possible because the sum contains three even, or two odd and one even numbers, so the product always will be an even number. Also, it cannot be 111111, and never can be an odd number if the sum is even".

Summary of results

The younger students needed and greatly benefitted from more guidance. The use of open problems and worked examples with these younger students was highly desirable in order to foster their learning. The generalization, with the right arguments, was successful only for upper secondary school students.

5. Conclusions

1. The most important step in mathematical transgressions is taking the first step. Mathematics educators are strongly influenced by research mathematicians. In Hungary, this is particularly true due to Hungary's tradition of producing so many world-famous research mathematicians. I am not saying that we do not need to hear them, I am only stating that we need to take the results of other disciplines, such as cognitive psychology, pedagogy, philosophy, linguistics, and neuroscience into consideration.
2. We must accept that there are controlled, randomized experiences in other science domains relating to mathematics education as well. The main message from these domains is that the mathematics learning and problem solving of experts are hugely different for novices. Most of the students need more guidance; discovery, problem-based, inquiry learning is not relevant for them. We must accept the characteristics of cognitive architecture and the result of comparative experiments between guided and unguided teaching.
3. My short experiments relate first of all to opening the closed problems. These forms helped to reach more students. Hungary's mathematics education culture has strongly supported the use of closed problems. This is based on its research tradition, as well as the arguments that discovery provides the best form of learning mathematics and that the students' mathematics achievements cannot be assessed by open problems. One of the consequences of this thinking is that text books and task collections only contain closed problems.
4. An important lesson from these other scientific domains is that the cognitive processes involved in the mathematics learning and problem solving of experts is different from those of novices, and most students need more guidance. Comparative experiments on the efficacy of guided versus unguided instruction and my own research show this as well.
5. While closed problem-based teaching may be effective in reaching the top tier of gifted students, it does not reach all promising students. Reaching all of the promising students and providing them with a solid mathematics education should be our goal as mathematics educators.

6. To be truly effective, this transgression needs to go further. The ideas put forth in this paper need to be utilized not only in mathematics teacher training, but also in ongoing mathematics education practice. This will make Hungarian mathematics education more effective at all levels.
7. We shall prepare our students for this teaching culture. The practicing teachers should also be prepared by in-service teacher training courses and through the open media.

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Mathematics, Arts and other Sciences

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Paradox resolution as a didactic tool

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Abstract. This paper contains a few reflections concerning a rather thankless task: teaching mathematics to the students of humanities. Our series of lectures *Puzzles* is a collection of mathematical and logical puzzles, conundrums, brain teasers, etc. It is thought of as a training in efficient problem solving. Below we give a few examples of puzzles from our lectures. Special emphasis is put on those problems which bear a touch of paradox. We add a few reflections on mathematical intuition and on pathology in mathematics.

1. Goal

It is simple to formulate our goal in general terms but its further specification requires a few comments. First of all, we are discussing *mathematical puzzles* and not standard mathematical exercises.

What is a difference between a math puzzle and a typical exercise? Usually, exercises are thought of as skill-improving tasks. After calculating, say, two hundred integrals you may get an impression that you know how to deal with integrals. Exercises are necessary to get fluency in the considered domain. We do not require a high level of sophistication in the formulation of exercises: they should be formulated as simply as possible, clearly presenting the problem to be solved. Exercises are thus the source of confirmation of mathematical intuitions accepted so far.

We think that mathematical puzzles should possess two properties. First, each puzzle should contain an intriguing plot, it must be interesting as a story which demands your attention and which causes a kind of cognitive discomfort in your mind. As a consequence, mathematical

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puzzles are often connected with real-life phenomena, though the mathematics behind them may be very sophisticated. Second, the solution of the puzzle should be surprising, unexpected, demanding reflection. Most valuable are puzzles whose solutions force us to correct our beliefs based on intuition only. Many such puzzles contain paradoxes whose solutions enable us to modify of mistakenly held beliefs.

We claim that paradox resolution is very instructive as far as the development of correct mathematical intuitions is concerned. Obviously, one should use several standard (normal, typical, natural) exercises in teaching mathematics – they doubtlessly serve as proper tools for stabilization of intuitions. However, to see clearly the limitations of our mathematical intuitions we should investigate also the objects which – for several reasons – are called *pathological* in mathematics. Such objects usually become later domesticated, thus leading to new mathematical domains.

The main goal of our lectures was to convince the students that solving math puzzles might be entertaining and instructive. The lectures were also thought of as a training in solving (abstract as well as very practical) problems with just a little help of mathematical reasoning. Judging from students' activity during the course and from the final essays they wrote, we may risk to say that the lecture was not a complete failure.

We think that an active participation in the course may be useful in development of critical thinking ability which is of great value for itself. In order to solve a puzzle you are supposed to be creative and not only to follow, say, a prescribed algorithm. The search of solutions is sometimes much more important than the final solution itself. One should consider the final solution to the investigated puzzle as a reward for intellectual activity engaged in the process of solution. The feeling of individual success is the best motivation for mathematical education.

2. Methodology

There exists a huge literature on *Mathematical Problem Solving* (MPS, for short). Everybody mentions of course the classical works of Polya in this respect (Polya 1945, 1954, 1965). We personally appreciate also the directives contained in the works of Schoenfeld (cf. e.g. Schoenfeld 1992).

The well known Polya's schema of mathematical problem solution consists of four steps: (1) Understand the problem; (2) Devise a plan; (3) Carry out the plan; (4) Look back. At each step there are more specific tasks which should be pursued. We are not going to report on them here, assuming that this classic approach is well known.

Schoenefeld's proposals stress, among others, the role of metacognition in the process of problem solving. He also suggests that the students should always attribute sense to the mathematical material they are searching.

We openly confess that we are hoping to elaborate our own methodology of solving problems using mathematical methods. Of course, we are not looking for universal solutions: we limit ourselves to the work with adult students who have bad memories from school and who are, in general, suspicious with respect to mathematics or even openly hate it. Thus, we are looking for therapeutic methods for such frightened victims.

Observing the students' activity during our course we have noticed that it is much more easier for them to acquire small concise chunks of dissipated knowledge rather than to listen to lengthy expositions of whole theories only accidentally illustrated with examples. We are of course aware of the fact that this kind of teaching does not provide an alternative for a full course on a given topic. However, our main goal is a training of solving problems with the help of mathematical methods and we strongly believe that students which become interested will be eager to read the textbooks containing full exposition of corresponding theories.

3. Material covered by the course

The puzzles are grouped into more or less homogenous topics. These are:

The Infinite	Numbers and magnitudes
Motion and change	Space and shape
Orderings	Patterns and structures
Algorithms and computation	Probability
Logic (paradoxes, sophisms, paralogisms)	Scientific, linguistic, philosophical puzzles

The grouping is tentative – some puzzles may belong to more than one group. We stress the unity of mathematics. Division of puzzles into groups is subordinated to didactic aims. After discussing the puzzles of each group we present short commentaries about the origin and function of the mathematical concepts involved in solutions of these puzzles. The historical comments are limited to a minimum. More attention is paid to discussion of main mathematical notions, methods, proof-techniques, types of reasoning, etc.

Currently we have collected about 120 puzzles with solutions and commentaries. Most likely, we will publish the collection with approximately 200 puzzles. Let us marginally mention that we have also translated (from English to Polish) some collections of logic puzzles (Smullyan 1982, 1987, 2009, 2013).

Mathematical puzzles have a long history. Actually, it might well be the case that the origins of mathematics are rooted in the efforts of puzzle solution at the time when no systematic mathematical knowledge had yet been collected. Puzzles served sometimes also as seeds of new mathematical disciplines.

There is a vast literature concerning mathematical puzzles (a few of our favorite collections are: Barr 1982, Gardner 1994, 1997, Havil 2007, 2008, Levitin, Levitin 2011, Mosteller 1987, Petković 2009, Winkler 2004, 2007). Recently one can find thousands of math puzzles in the internet. One should also mention several mathematical competitions, either national or worldwide.

4. Examples

Many of the puzzles presented during the course concern several aspects of infinity. We think that they are very instructive for developing mathematical intuitions. The students should become aware that mathematics transcends physical reality and it is not an approximation of this reality only.

We encounter infinity in several contexts in mathematics, e.g. infinitely large objects (for instance, infinite sets), infinitely small objects (for instance, infinitesimals in non-standard analysis), infinitely complex objects (for instance, fractals), limits as objects obtained in an infinite process, objects "at infinity" (points, lines, etc.), infinite sums and products. We try to "domesticate" these notions via specially prepared puzzles. They concern e.g., the harmonic series, several *supertasks* (Thomson's lamp, Laugdogoitia's balls, ancient paradoxes of motion, etc.), methods of constructing fractal objects, Hilbert's hotel, spirals, infinite binary tree, and so on.

4.1. Sum-product puzzle

S knows only the sum and P knows only the product of two numbers x and y and they are both aware of these facts. They both know that $x > 1$, $y > 1$, $x + y \leqslant 100$. The following dialogue takes place:

– P: I do not know the two numbers.
– S: I knew that you did not know them.
– P: Now I know these numbers.
– S: Now I know them, too.

Find these numbers x and y. This is Freudenthal's puzzle from 1969, popularized later by Gardner (Freudenthal 1969, 1970, Gardner 1979). The above statements (in this order) imply several arithmetic facts which enable us to find the (unique) solution: the numbers in question are 4 and 13.

4.2. Bouncing balls

Let the mass of two balls be M and m, respectively. Suppose that $M = 100^n m$ ($n \geqslant 0$). We roll the ball M towards ball m which is near the wall. Thus M hits m which bounces off the wall. How many collisions occur (jointly, i.e. between M and m and between m and the wall) before the ball M changes direction? The answer depends on n, of course.

A surprising fact is that the number of balls' collisions is equal *precisely* to the first $n+1$ digits of π. It is worth noticing that the result is purely *deterministic* and not based on probability, as in the well-known Buffon's needle puzzle. The solution is presented in Galperin 2003.

4.3. Sliding ladder

A ladder of length L is leaning against a wall. The bottom of the ladder is being pulled away from the wall horizontally at a uniform rate v. Determine the velocity with which the top of the ladder crashes to the floor.

Let x be the distance of the bottom of the ladder from the wall (thus measured on the ground level) and y be the distance between the bottom of the wall and the point at which the ladder touches the wall.

It seems natural to use the Pythagorean Theorem: $x^2 + y^2 = L^2$. Further, we have $\frac{dx}{dt} = v$, and hence $\frac{dy}{dt} = -v \cdot \frac{x}{y}$ (which we obtain by differentiating the equation $x^2 + y^2 = L^2$ in which both x and y are functions of time t). Thus, $\frac{dy}{dt} \to \infty$ when $y \to 0$. Assuming that the top of the ladder maintains contact with the wall we obtain an absurdity: the velocity in question becomes infinite!

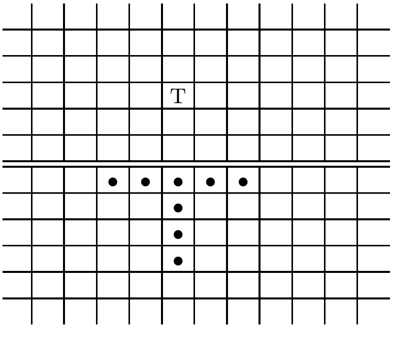
Actually, at a certain moment the ladder looses contact with the wall. After that, the motion of the ladder is described by the pendulum equation. The solution of this puzzle can be found in (e.g. Scholten & Simoson

1996). More accurate descriptions of this problem involve friction, pressure force, etc.

4.4. Conway's army

The game is played on an infinite board – just imagine the whole Euclidean plane divided into equal squares and with a horizontal border somewhere. You may gather your army of checkers below the border. The goal is to reach a specified line above the border. The checkers move only vertically or horizontally. Thus diagonal moves are excluded. As in the genuine checkers, your soldier jumps (horizontally or vertically) over a soldier on the very next square (which means that he kills him) provided that it lands on a non-occupied square next to the square occupied previously by the killed soldier.

It is easy to show that one can reach the first, second, third and fourth line above the border. As an example, here is a minimal army capable of reaching the third level:



However, no finite amount of soldiers gathered below the border can ever reach (by at least one surviving soldier) the fifth line above the border! The solution can be found (e.g. Berlekamp, Conway & Guy 2004; Havil 2007). The solution uses a representation of the army as a formal polynomial. It is invented in such a way that the rules of the game preserve its value. The target is given a specific value and one can show that the target's value cannot be reached by any finite army below the border. There exist several generalizations of this game with their own limitations as far as the accessible level above the border is concerned.

4.5. Balls in a box

Suppose you have an infinite number of balls, more exactly: you have an infinite number of balls numbered with 1, an infinite number of balls numbered with 2, an infinite number of balls numbered with 3, etc. – an infinite number of balls numbered with any positive integer. You have also a box, in which at the start of the game there is a certain finite number of such numbered balls. Your goal is to get the box empty, according to the following rule. At each move you are permitted to replace any of the balls inside the box by an arbitrary finite number of balls with numbers less than the number on the ball removed. Of course, balls with number 1 on them are simply removed from the box, because you can not replace them by balls numbered with a positive integer smaller than 1. Is it possible to make the box empty in a finite number of steps?

Everybody immediately can see the trivial solution: just replace each ball in the box with a ball numbered by 1 and then remove all these balls, one by one. Thus, the answer to the puzzle is certainly affirmative. However, there is a subtlety in this puzzle. It is the fact that you can not *in advance* predict the number of steps required to finish the game. The solution is described e.g. in Gardner 1997.

The game can be represented by a tree. The root of the tree represents the empty box. Its immediate successors represent balls in the box at the beginning of the game. If you remove, say, the ball with numer n from the box and replace it by, say, k balls with number m ($m < n$), then from the node occupied previously by n you draw k edges to new leafs of the tree, all labelled with the number m. Each path in the tree corresponds thus to a (finite!) decreasing sequence of positive integers. Removing a ball labelled with 1 means removing the corresponding node in the tree.

The proof that the game always leads to the empty box uses the *König's lemma*, which says that an infinite, finitely generated tree contains an infinite path. Recall that the tree is *infinite* if it has an infinite number of nodes and it is finitely generated if each node has only a finite number of immediate successors.

Suppose that the tree of the game is infinite. It is of course finitely generated, according to the rules of the game. Hence, by König's lemma, it contains an infinite path. But each path of the tree is a *decreasing* sequence of positive integers and therefore can not be infinite – the set of all positive integers is well ordered by the usual less-than relation. We got a contradiction, so the supposition that the tree of the game is infinite should be rejected.

There is a bloody version of this puzzle, concerning Herakles killing a hydra. There are also very serious and important theorems behind the puzzle. Namely, it can be shown that some sentences in the language of the first order Peano Arithmetic (think of the arithmetic you know from the school, that is enough) though true in the standard model of this theory (i.e., roughly speaking, true statements about the genuine natural numbers) are nevertheless unprovable in this system. They are provable only in a much more stronger system, where infinitary tools are allowed.

4.6. Ant on a rubber rope

This cute puzzle has several versions (cf. e.g. Graham, Knuth & Patashnik, 1994), a typical one being the following:

> An ant starts to crawl along a taut rubber rope 1 km long at a speed of 1 cm per second (relative to the rope it is crawling on), starting from its left fixed end. At the same time, the whole rope starts to stretch with the speed 1 km per second (both in front of and behind the ant, so that after 1 second it is 2 km long, after 2 seconds it is 3 km long, etc). Will the ant ever reach the right end of the rope?

It should be stressed that this is a purely mathematical puzzle – we ignore the ant's mortality, we assume that there exist infinitely elastic ropes, etc.

People usually doubt that the ant could achieve the goal in a finite period of time. However, the answer is affirmative – the ant certainly will reach the right end of the rope, though it takes a really long time interval.

The dynamic aspects of the problem may cause some troubles in its solution. In general, one should solve a (rather simple) differential equation describing the motion in question. However, one can approach the problem also in a discrete manner, as follows.

The main question is: which part of the rope is crawled by the ant in each consecutive second? It is easy to see that:

During second	the ant crawls	part of the whole rope
first	1cm out of 1km	$\frac{1}{100000}$
second	1cm out of 2km	$\frac{1}{200000}$
third	1cm out of 3km	$\frac{1}{300000}$
n-th	1cm out of $n$km	$\frac{1}{n \cdot 100000}$

Hence the problem reduces to the question of existence of a number n such that:

$$\frac{1}{100000} + \frac{1}{200000} + \frac{1}{300000} + \ldots + \frac{1}{n \cdot 100000} \geqslant 1.$$

This is of course equivalent to the existence of n such that:

$$1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}\geqslant 100000.$$

We know that the *harmonic series* $\sum_{n=1}^{\infty}\frac{1}{n}$ is divergent. Therefore, there exists a number n such that $1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}\geqslant 100000$. This number is really huge, it equals approximately $e^{100000-\gamma}$, where γ is the Euler-Mascheroni constant:

$$\gamma=\lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k}-\ln n\right)=0,5772156649501\ldots$$

This constant remains a little bit mysterious – for instance, we do not know at the present whether it is rational or irrational.

Essential in this puzzle is the fact that the considered velocities are constant. If, for instance, the rope is doubled in length at each second, then the poor ant has no chance to reach the right end of the rope (crawling, as before, with constant speed). The puzzle has also interesting connections with the recent views concerning the Universe. Remember: the space of the Universe is expanding, but the speed of light is constant. What are the consequences of these facts for the sky viewed at night in a far, far future?

5. Intuition and pathologies

Shaping of proper (correct, adequate) mathematical intuitions is declared as the main didactic goal in math teaching. Independently of the question *how* it should be done there arises a very natural question: which intuitions are proper (correct, adequate)? Who should be responsible for establishing an allegedly complete list of mathematical intuitions, or more modestly, intuitions which are considered as proper just currently? It seems that there is no easy answer to that question.

5.1. What are mathematical intuitions?

Mathematical intuition is presented in the axioms, which is evident: they are accepted without proof, on the basis of some intuitive beliefs alone. One should keep in mind, however, that in the most important mathematical domains axiomatic approach was preceded by a huge cumulation of knowledge about the domain in question. The literature on this subject is very large. Let us only mention that e.g. (Feferman, Friedman, Maddy & Steel, 2000) try to answer the question of whether mathematics needs new axioms.

We meet mathematical intuition also in the everyday practice of mathematicians – in reasoning by analogy, in making generalizations, in several heuristic rules of thumb, in using inductive assumptions before formulation of a hypothesis to be proven, etc. Needless to say, the main body of mathematical education (especially teaching on the elementary level) is based on intuitive explanations.

In our opinion, simple mathematical intuitions emerge from two sour-Ssces: our cognitive powers and symbolic violence of the school. The first source is investigated by cognitive sciences, reflections on the second belong to educational studies. More sophisticated intuitions are obtained during creative work of professional mathematicians.

School programs of teaching mathematics, as far as we know, contain first of all several algorithms whose familiarity is necessary in practical calculations, simple planning tasks, measuring, etc. Talking about mathematical notions, their mutual dependencies, their origin, etc. takes place only occasionally. The role of proofs is also diminished, as we have noticed from recent school textbooks. One may get thus an impression that mathematics means first of all calculations with the help of prescribed algorithms and once and for ever established formulas, treated as dogmas.

Professional mathematicians will laugh at such impressions. Mathematics means rather a play with ideas, a game whose rules are determined by laws of reasoning. Obviously, these laws are not completely arbitrary but are based on logic and entailment.

We support the view that mathematics is: search for patterns, solving problems, making conjectures, proving hypotheses. Thus, we think of mathematics as of an activity. The results of this activity, i.e. books and papers written by mathematicians do not exhaust the whole of mathematics. They should be prepared according to the standards accepted by the mathematical community. However, the published text is only an iceberg of the entire totality of mathematical activities involved in its preparation.

We propose to understand mathematical intuition in the following way. First, let us point to *elementary* (*primary*) intuitions which are somehow (we are not going to discuss how – let this problem be investigated by cognitive sciences) connected with forming such concepts as number, measure, distance, ordering, etc., i.e. concepts which belong to mental categorization of every day experiences of humans. This sort of cognition very likely is structured by evolution and should be investigated in an empirical way. Next, let us point to *secondary* (*acquired*) intuitions which evolve in the process of learning mathematics. The formation of these intuitions is influenced mainly by the *symbolic violence* in the school. At the very beginning of the study of mathematics the arguments from authority (of the teacher, textbook, sources used in teaching) play an important role. Good solutions of the problems in the class are prized, bad solutions are corrected. In this process the pupils are encouraged to change their attitude: from passive observation to active behavior in searching solutions. At this stage, argumentation from analogy may appear helpful, we think. Finally, let us point to the *advanced* (*complex*) intuitions of professional mathematicians who are *doing* (creating) mathematics by themselves. Such intuitions are beliefs based on the subject's own experience. They may be of course strongly influenced by the subject's previous knowledge of mathematics, his skills, the discussions with others, written tradition, etc. We think that these intuitions can be verbalized and hence are accessible for investigation. Moreover, such an investigation is not restricted to introspection. Rather, we suggest that revealing professional mathematicians' intuitions should be based on the analysis of the source texts.

Unlike the more-or-less stable intuitions connected with everyday experience, mathematical intuition is more dynamic. Major sources of changes of mathematical intuition seem to be: paradoxes, scientific programs, new results in mathematics. Without going into details let us only add that mathematical intuition is also influenced by (among others): aesthetic values, empirical experiments, and mathematical fashion.

5.2. Pathological objects

The very term *pathological* immediately implies some negative associations. On the contrary, mathematical objects named *pathological* (sometimes also: *paradoxical*) are signs of strength and vitality of mathematics.

There seem to be at least two typical situations in which one speaks about pathological objects in mathematics:

1. *Unexpected objects, causing a clash with established intuitions.* At a given epoch, mathematicians share intuitive views about the concepts they are dealing with. Discoveries of new kinds of objects (e.g. negative, imaginary, irrational numbers) may contradict these intuitions. But if the new objects appear fruitful in applications, if they are equipped with a sound theory, then the initial intuitions are forced to change.
2. *Pathologies constructed specially, on purpose.* Such objects are introduced consciously, with specific goals in mind. They may show

> the relevance of assumptions accepted in some theorems or show explicitly the range of some mathematical concept. For instance, when one proposes new definitions of concepts formerly understood in an intuitive way it may happen that this new precise definition does not exclude some "monsters", as it was the case with a general definition of function. There are many examples of such artificially created pathologies in analysis, measure theory and general topology.

The important fact is that pathologies become *domesticated*. A prototypic example of an object originally thought of as a pathological one and then becoming normal, standard, "domesticated" is the Cantor set. Recently no one among professional mathematicians considers it as a pathology. This is due to its fundamental role in e.g. topology. Only in popular books authors try to frighten the innocent readers with devilish mysteries of the Cantor set and other fractal sets.

It is ridiculous – in our opinion at least – to expect that our common intuitions connected with everyday experience should be respected at any stage of sophisticated mathematical constructions. At any rate, there are cases in which operating with well known objects, using perfectly natural proof techniques one arrives at results which are strongly divergent from common intuitions – cf. eg. Smale's theorem about sphere's eversion or constructions involving exotic spheres.

The concept of pathology in mathematics is pragmatically biased and relative to the development of mathematical theories. There are no "absolutely pathological" objects. Similarly, the concept of a "well-behaving" object is context dependent and connected with applications under consideration.

In some of the puzzles presented during our course we try to familiarize students with some celebrated mathematical pathologies.

6. Which paradoxes are useful in teaching math?

Teachers of mathematics are supposed to be careful in talking about their subject: the students should be encouraged to learn new topics but not become frightened by them. We are trying to follow a secure (in our opinion at least) way of introducing the methods of mathematical problem solving to our students. Thus, we ask ourselves the following questions: (1) At which moment may we introduce a paradox? (2) How to explain what is standard in the investigated domain? (3) Which parts of the MPS-methodology are applicable in the case of normal (standard, natural) objects and which of them are suitable for problems and paradoxes concerning pathological objects? At the moment, we are collecting our observations and we do not want to formulate any hypotheses or conjectures now.

Nice inspirations for the creation of math puzzles may come from theorems which have a touch of paradox. Thus, the following facts, theorems, constructions, curiosities, seemingly paradoxical results, etc., could be used for compilation of entertaining math puzzles: Smith-Cantor-Volterra sets, Weierstrass function, Dirichlet and Thomae functions, Peano and Hilbert curves, Alexander's horned sphere, Wada lakes, Knaster's curve.

There exist books devoted entirely to counterexamples in different domains of mathematics (e.g. Gelbaum, Olmsted, 1990, 2003; Klymchuk, Staples, 2013; Steen, Seebach, 1995; Wise, Hall, 1993).

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Urszula Foryś

Interdisciplinary character of mathematics: biomathematical perspective

Abstract. This article presents the author's point of view on the situation of interdisciplinary research in Poland in the context of biomathematics. Some well established examples of mathamatical models in biology and medicine are described. Some bibliographical positions available in Polish on that topic are listed.

1. Introduction

Mathematics constitutes a kind of language which is used for ages by people to describe surrounding world. Even in ancient times many mathematical notions we use today were known. As an example we can mention golden ratio, which was described by Euclid in his *Elements*. Development of complex mathematical tools is inextricably linked with the development of other sciences, which has been especially visible when modern physics started to develop. Such standard notion as derivative has its roots in physics, as it just reflects a speed in some motion.

Nowadays, mathematics is present in every type of our activity. Without mathematical models, development of better cars, faster trains, modern planes, safe usage of internet banking, new diagnostic methods of various diseases etc. would be not possible and our world would look completely differently. However, there is a very limited number of people who realise that mathematics plays so important role in modern world. Typically, when someone hears something about mathematics, his/her usual reaction is to think "that's horrible".

Biology and medicine are those scientific disciplines in which mathematics is used, but this usage is not so wide as could and should be. This

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Key words and phrases: biomathematics, biomedical applications, mathematical model, dynamical system, interdisciplinary research.

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article is devoted to the topic of mathematical modelling in biomedical sciences, its history and perspectives, from the point of view of Polish biomathematician, who I am.

2. Historical outline

Since ancient times people tried to use mathematical language (abstract mathematical notions) to describe various natural phenomena. Ancients applied mathematics in astronomy and geometric optics. The oldest mathematical model of some biological phenomenon is probably the Fibonacci sequence. It first appeared in the book *Liber Abaci* (1202) by Leonardo of Pisa, known as Fibonacci. In this book Fibonacci considered the growth of an idealised rabbit population under the following assumptions. A newly born pair of rabbits are put in a field where they are able to mate at the age of one month so that at the end of its second month another pair of rabbits appears. A mating pair always produces one new pair every month from the second month on. The puzzle that Fibonacci posed was: how many pairs will there be in one year? As an answer he listed the number of pairs at the end of each month:

$$1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ 34,\ 55,\ 89,\ 144.$$

More precisely, at the end of the mth month, the number of pairs of rabbits is equal to the number of new pairs (which is the number of pairs in month $m-2$) plus the number of pairs alive last month $m-1$. This gives the mth Fibonacci number:

$$F_m = F_{m-1} + F_{m-2} \quad \text{for} \quad m > 2,$$

which can be calculated knowing the initial data $F_1 = F_2 = 1$. What is interesting, this sequence is directly related with golden number, as the ratio F_{m+1}/F_m tends to this number [1]. In such a way mathematical tools, specifically dynamical systems, was applied in biological sciences.

For many years development of the branch of science called now biomathematics has not been observed. Only at the end of the nineteenth century English demographer Thomas Malthus published *An Essay on the Principle of Population* in which he observed that sooner or later population will be checked by famine and disease, leading to what is known as a Malthusian catastrophe. He pointed out that the increase of the number

1. Clearly, if $x_m = F_{m+1}/F_m$, then $x_m = 1 + 1/x_{m-1}$, and the limit satisfies $g = 1 + 1/g$, that is $g^2 - g - 1 = 0$ implying $g = \left(1+\sqrt{5}\right)/2 = \Phi$.

of people is governed by the geometric progression, while the amount of food increases only according to the arithmetic progression [2]. We again found mathematical tools useful to explain the problem from population dynamics. And again discrete dynamical system was used in this field. What is interesting, nowadays the notion of Malthus model is associated with continuous equation

$$\dot{N} = \frac{dN}{dt} = rN, \tag{1}$$

where $N(t)$ is the number of people at time t and $r > 0$ is the birth rate (in general, it could happen that $r < 0$) [3].

For many years scientists discussed the hypothesis of Malthus, and the final solution was found by Pierre Francois Verhulst in the middle of eighteen century. Verhulst proposed so-called logistic model in which he took into account carrying capacity of an environment, and this carrying capacity is the result of bounded resources of the specific environment. Under the assumption of bounded capacity competition of the resources must appear, and this inhibits exponential growth of the population living in this environment. In the logistic equation the birth rate r is not constant, but depends on the population size, such that instead of Eq. (1) we have

$$\dot{N} = r_f(N)N,$$

where r_f is decreasing, $r_f(N) = r(1 - N/K)$, $K > 0$ reflects carrying capacity. This is the begging of application of continuous dynamical systems in population dynamics.

We should notice that the equation $\dot{x} = ax$ is the universal model for many natural phenomena, especially for $a < 0$ it reflects the process of degradation of biochemical substances, and is applied e.g. for estimation of drug dosage (however, in most cases medical doctors do not know about it), or for determining the age of an object by using the radiocarbon dating.

For interested reader I would like to refer to (Foryś, 2005; Murray, 2002; 2006; Rudnicki, 2014) for further information on simple mathematical models in biomedical applications.

2. Clearly, if $N_{n+1} = qN_n$, $q > 1$, describes the number of people, while $X_{n+1} = X_n + r$, $r > 0$, reflects the change in food resources, then we have $N_n = q^n N_0$ and $X_n = X_0 + rn$ which gives $X_n/N_n \to 0$, and the convergence is vary fast.

3. Notice, that geometric sequence $N_n = q^n N_0$ and exponential function $N(t) = N_0 \exp(rt)$ take the same values at $t = n$ for $q = \exp(r)$, such that both models reflect the same type of the growth.

3. Closer to the present time

In the twentieth century mathematical models appeared in biological applications, especially in the description of interacting populations, more often. The oldest and probably best known model of that type is a prey-predator model known as Lotka-Volterra model, proposed in the twenties of twentieth century. It is also the model which was a spectacular success in explaining some ecological phenomenon. The model is described as a simple system of ordinary differential equations that reads

$$\begin{aligned} \dot{V} &= rV - aVP, \\ \dot{P} &= -sP + abVP, \end{aligned} \tag{2}$$

where $V = V(t)$ reflects the size of prey species and $P = P(t)$ stands for predators, in the context proposed by Vito Volterra, who used it to explain seeming paradox regarding fishery in Adriatic Sea after the First World War [4]. On the other hand, Alfred Lotka proposed the same system of equations to describe hypothetical biochemical oscillatory reaction on the basis of mass action law. It turned out that the same model was a basis to formulate ecological law of mean values preservation, as well as was able to predict existence of oscillatory chemical reactions. At present, ecologists and chemists know it very well, however typically do not realise that this knowledge has mathematical origin (c. f. Foryś, 2005; Foryś, Matejek, 2014; Murray, 2002; 2006; Rudnicki, 2014).

Polish scientists also have a big success in the field of application of specific model in biomedical sciences. Professors Maria Ważewska-Czyżewska and Andrzej Lasota worked together on modelling of red blood cells production, and after proposing their mathematical model they were able to develop a new therapeutic method called "oxygen tent" (c. f. Lasota, Mackey, Ważewska-Czyżewska, 1981; Ważewska-Czyżewska, Lasota, 1976, Ważewska-Czyżewska, 1981).

Unfortunately, it should be noticed that there are not many biologists or medicals similar to prof. Ważewska-Czyżewska who was not afraid talk with mathematicians. In most medical doctors and biologists are scared of mathematics, and it is not only Polish speciality. However, I do not know details of education in different countries, but I know well that mathematical education for students of biology and other similar fields of study

4. It could be shown that all solutions of Eqs. (2) with positive initial data have the same mean values $(s/ab, r/a)$. These values changes to $((s+c)/ab, (r-c)/a)$ when individuals are caught with intensity c. Therefore, fishing/hunting results always in decreasing of predator population and increasing prey population level, which was very surprising for ecologist 100 year ago, and was explained using mathematical model.

leaves a lot to be desired. Students of such fields treat all mathematical courses like a necessary evil, and forgot everything they learned to very fast. Educational gaps result in difficulties in interdisciplinary cooperation, while such a cooperation is inevitable if we want to be competitive and innovative.

My own experiences in interdisciplinary cooperation in Poland are rather negative. It has started during my PhD studies. My supervisor, Prof. Wiesław Szlenk had a colleague being an immunologist, and he worked with him on mathematical modelling of immune reaction. I was also partially involved in this project, and I was able to become convinced how difficult finding a common language could be. After the death of my supervisor, who unfortunately died twenty years ago, continuation of the cooperation was not possible, because it was based on personal relationships of Prof. Szlenk and his colleague. I have tried to establish interdisciplinary cooperation many times, but I failed. For a short time, at the end of the nineties of the twentieth century, I worked together with foresters from the Forest Research Institute on the project concerning estimations of specific pests gradations in Polish forests. However, just after finishing the project, our cooperation also finished. Recently, I have started cooperation with psychologists on modelling of marital interactions (c.f. Bodnar, 2014; Gottman, Murray, Swanson, Tyson, Swanson, 2002; Murray, 2002; 2006) for introduction to such type of modelling) and I hope this cooperation will develop. However, my real interdisciplinary work is not settled in Poland. I am lucky to have a long time cooperation with the Institute for Medical Biomathematics, Bene Atharot, Israel, led by Prof. Zvia Agur. In the IMB, biologist works together with chemist and mathematicians, and Prof. Agur takes care of they common work that should come to some sort of arrangement. The atmosphere at the institute is really great, and I am very glad that my skills can be usefully used, although in Israel, not in Poland. I am afraid that in Poland for a long time it will not be possible to create such institutes.

On the other hand, I know some examples of Polish interdisciplinary teams, but the role of mathematics is typically marginal there. More precisely, mathematical models are used for persons coming from different fields, but they either do not try to make any mathematical analysis that could bring some new insight into the knowledge of the analysed process. However, it is not only Polish problem. There are other evidences that mathematics is treated marginally, because sometimes proposed models are not proper or incorrect analysis gives wrong results, what is even worse comparing to not making analysis [5]. In such a context it is obvious

5. I can give one simple example. In (Bratsun, D.; Volfson, D.; Tsimring, L.;

that mathematicians must be involved in any interdisciplinary projects where mathematical modelling is used to avoid various kinds of errors that could appear. This is strongly correlated with financial supports of such type of research. Unfortunately, in Poland founds for basic research are divided into separated fields and it is almost impossible to get financial support for interdisciplinary projects. On the other hand, the institution that should support such projects, that is National Center for Research and Development (NCBiR) gives support mostly projects that are of industrial interest, and that could find direct applications on the market, not such research that could help to understand some phenomena and could give some practical results only after many years. It is obvious that such type of research, like e.g. in the field of civilisation diseases, including cancer (which is of my main interest), could not be supported by industrial companies. One can think that this is not true, because pharmaceutical companies should be interested in supporting this type of research. However, it should be marked once again that we are talking about the basic research that is not focused on direct applications, and therefore I am not able to imagine a company which will be interested in it.

Despite financial problems, creating an interdisciplinary team is always difficult due to communication problems, as I have mentioned before. Scientists from the older and middle generations are typically self-contained in their fields and they are not able or/and do not want communicate with persons from different fields. I hope that the young generation will be able to change this situation. Some of our former students devoted themselves to interdisciplinary research and have success in their work. After defending their master's theses in mathematics, they turn to more biological/medical problems and got PhD in different fields, which means that they are able to communicate with biologists or/and medical doctors. Dr. Zuzanna Szymańska defended her PhD in biology and work in Interdisciplinary Center of Mathematical Modelling at the University of Warsaw. Dr. Joanna Stachowska-Piętka and Dr. Jan Poleszczuk defended their PhD theses in technical sciences. Joanna works in Nałęcz Institute of Biocybernetics and Biomedical Engineering and her research is related to peritoneal dialysis, while Jan got post-doc position in Moffit Cancer Center in the USA. This is an interdisciplinary center in which scientists

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Hasty, J., 2005) the authors proposed simple linear model with time delay to reflect some biochemical reaction with delayed degradation. They used the same idea as for the model without delay. However, they do not noticed that introducing delay leads to negative solutions, which is of course biologically irrelevant. Therefore, in (Miękisz, J.; Poleszczuk, J.; Bodnar, M.; Foryś, U., 2011) we proposed corrected model which could be used in such a case.

from various fields work together on finding effective methods of cancer treatment, but it will be not possible without huge US government support. I hope that he will be able to use this experience in starting similar project in Poland, and I will be able to participate in it.

4. Bibliographical remarks

In this short article I have only noticed some important positions of bibliography. However, I would like to mention that EU financial support allowed for publishing three monographs devoted to various applications of mathematics (Bartłomiejczyk, 2013; 2014; 2015), where many interesting problems were described and explained in mathematical language. There are also other text-books available in Polish focused on mathematics in biology, like (Rudnicki, 2014; Wrzosek, 2010). However, the text-book by D. Wrzosek (2010) was prepared for students of biology, while the text-book of R. Rudnicki is much more advanced.

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Maria Gokieli, Marcin Szpak

Visualization and Experiment in (School) Mathematics

Abstract. We want to argue for teaching mathematics with computers, treated as tools for visualizing and experimenting. We will build upon our didactical experience related to:

– a course of mathematics for a broad audience of school students at the University of Warsaw („Matematyka dla Ciekawych świata");
– courses at the Interdisciplinary Centre of Mathematical and Computational Modelling of the University of Warsaw;
– courses for teachers at the Cardinal Stefan Wyszynski University in Warsaw;
– teaching at the school and university levels in Poland and in France.

We will comment on the feedback from pupils, students and teachers. We think that software used by researchers on one hand, and common office packages on the other, can be successfully used in school didactics. We will give examples of using software like Mathematica, Matlab/Scilab, R, Excel in presenting mathematical notions, getting familiar with and as a proposition of integrated teaching of sciences.

We will also discuss risks related to teaching with computer tools.

Key words and phrases: mathematical software, computer, experiment.

AMS (2000) Subject Classification: Primary 97A80, Secondary 97U50, 97D40, 97D50.

Maria Gokieli is a mathematician and Marcin Szpak a computer scientist. They are both involved in the realization of *Matematyka dla Ciekawych Świata* project.

1. Introduction

The Polish curriculum defines, among others, the goals of school education. At its third and fourth levels (i.e. for students aged 12-15 and 16-19), for mathematics, we have five such goals: using and producing information (mathematical text, mathematical language), using and interpreting representation (mathematical objects, mathematical notions), mathematical modelling (situation and its mathematical model), using and forming a strategy (for problem solving), and finally reasoning and building an argumentation.

Among these goals, the first two concern the mathematical language, and the others deal with applying this language to a concrete situation. Even if both, language and applications, have obviously less and more difficult parts, and even if both may cause difficulties, it is clear that the mathematical language is a kind of bricks with which we can build a strategy for a concrete problem solving. In this sense the first two goals of the Polish curriculum are "low level" and the three others are "high level", when looking at their complexity.

It is also clear that there is a real difficulty for students for passing from one to the other. This difficulty has for a long time been a real obstacle for Polish students, that now seems to be gradually overcome, as the reports of the Program for International Students Assessment show (PISA, 2009) and (PISA, 2012). Still, the difficulty, that we would even call a "gap", clearly remains. We think that this gap can be diminished by an effort towards visualizing mathematical notions and towards more experimenting with mathematical ideas and models.

The examples and reflections that we are going to present in the paper are not fruit of a planned research in didactics, but rather of our – transgressive – teaching experience. It comes mostly from the courses named *Mathematics for Curious People (Matematyka dla Ciekawych Świata)*, which have been held at the University of Warsaw for seven years (see the web page: ciekawi.icm.edu.pl). They are addressed to pupils in two age groups: 13-15 and 16-19. Each year, a series of meetings has been proposed on a chosen subject, such as Mathematical Models, Infinity, Cryptology, Numbers, Pi, Languages, etc; each meeting consisting of a one hour lecture and two hours of exercises, together with an integration break. The goal is to deploy the young people interests in mathematics and computer science. Thus, the course is intended as something in between popularization and regular teaching. Its main principles are: going beyond the school program, a problem-oriented teaching, working in small groups with a lot of individual approach, melting various milieus and pupils with different knowledge levels, and finally reducing external motivation.

The effect is a growing group of participants (30 in 2007 against 200 in 2016), a live interest of school teachers and a promising, deep involvement of university students. In a word, a lot of new interactions related to teaching mathematics.

These courses are transgressive to the normal education streaming in many aspects: high school pupils in an academic environment, learning non–school subjects, teaching done in a large extent by university students, no rewarding and no pressure. We think that these specific conditions gave birth to some new ideas; they also gave time for experimenting with them, so that they can be further applied in traditional teaching. Let us share here some samples of these and discuss its possible effects.

2. Visualization

The advantages and the methods for visualizing mathematics have been described at many other points of the Conference (e.g. I. Lénart, J. Novotnà, A. Sondore, S. Turnau, P. Vighi, B. Wawrzacz, L. Zaręba). A survey of former studies of the role of visualization in mathematics education, which started in the 1970's, can be found e.g. in (Presmeg, 2006) and (Clements, 2014). Clements analyzes the possible meaning of the word (geometrical representation for mathematicians/mental representation for psychologists or cognitive scientists), and discusses methods for identifying visual and non-visual students. Presmeg invokes the research on many psychological and cognitive aspects of visualization. Indeed, when we address visualization from the point of view of its effectiveness in didactics, we come to the more fundamental questions in cognitive science. "Exactly what makes imagery effective in mathematics?", Presmeg asks in the conclusions of the cited paper (p. 227), pointing out that this remains an important, needed point of discussion, related to many other questions on cognition, thinking process, problem solving and finally teaching. But we think that this question should also be a point of reflection for teachers. How do we see visualization? Is it only a help tool, or rather a natural representation of notions? Or may it even be primary to mathematical notions? Our own conviction is that the answer is situated close to the third option, and that this is a natural reason for the importance of visualizing mathematics, that we discover when examining the learning process' effects.

It is clear that if we want to visualize and make it a method in teaching, we are faced with the possible use of the computer. The influence of visual aspects of computer technology on the dynamics of learning mathematics is one of the thirteen "Big research questions" in visualization for mathematics education, asked by Presmeg (2006). We would like to give here some examples for visualization in three different areas of mathematics (without geometry) and at three different school levels, hoping that they may give some ideas to the teacher. These examples involve the computer, but the use of the computer is not crucial there, it will appear only as a tool. The reader should judge its value in this context, in these concrete situations.

Example 1: Solving a linear system

Wolfram Alpha, the free version of Mathematica is available online. Fig. 1 shows the simplicity of using this software for a visual solution of a linear system. Of course, the exercise is certainly done at the blackboard in our classes, giving the same effect: illustrating the formal algebraic operations and their possible effect. We note that this illustration leads to abstraction: it gives all the possible results (the lines can intersect in

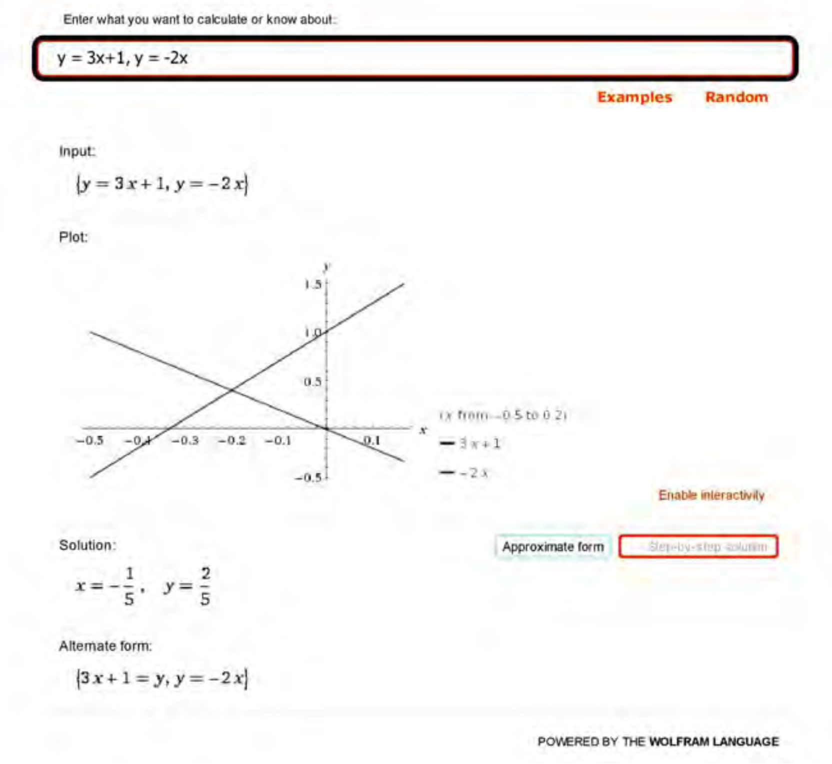


Figure 1. A basic computer tool for a graphical solution of a linear system.

one point, be parallel or be identical) and the major differences between them. It is available to anybody who is willing to imagine, even if deprived of any technical ability to manipulate symbols (yes, it is also a technical ability, like using computers, the main difference being maybe our pupils master it less than we do). Now, the important advantage of doing it by hand is certainly that we do not use any superfluous tool, and do not create the impression that a computer is necessary for solving the problem and drawing the solution. Also, we do not have to learn the technicalities of using a specific program. However, from the point of view of the young students, the use of the computer bypass all the technical problems that may arise for them (how to transform, how to draw the line etc.) and that may hide the idea itself. The computer, besides, gives an additional possibility of easily repeating the same operation, that is, experimenting – we will come back to this question in the next section.

Example 2: Trigonometry

What is striking in the polish curriculum in its all possible versions, is the absence of any mention about the trigonometric circle when teaching trigonometry [1].

This is, in our conviction, the cause of the difficulty the (good) polish students have afterwards to understand and use polar/spherical coordinates. This is also certainly the reason why trigonometry, in general, is not understood, but just taught mechanically. At the same time, there are dozens of animation available in the web, showing the definition of the trigonometric functions and how the graphics of these come up from these definitions – see some examples on Fig. 2 (you have to imagine or check the animation going). It is our experience that a few minutes of looking at this animation, commented by the teacher, makes a dramatic positive change in students' attitude towards trigonometry. It also help in learning solving trigonometric problems: angles are set much more naturally on a circle than on an axis.

Let us stress on the fact we take the teacher role here for granted. As noted by Mayer (1997, p. 18): "presenting an animation – however clever – without concurrent narration is unlikely to promote meaningful learning". We will come back to this question in Section 4.

1. Another question when speaking about trigonometry in polish high schools is why the students are not given any real application of trigonometry, apart from calculating lengths given the angles or angles given the lengths. This gives no idea of the fundamental importance of trigonometry in applications such us electricity, electronics, image processing etc., which, even if too difficult to learn at this level, should certainly be mentioned as a motivation to study the subject and could be included in form of a follow-up study. We will come back to this idea in Section 3..

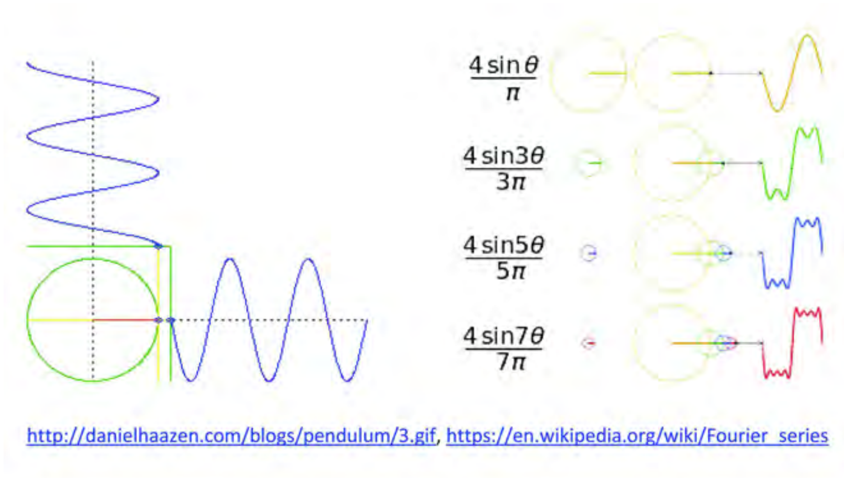


Figure 2. Trigonometric animations available on the web.

Example 3: Calculus

A much more advanced, but still a classical example of visualizing with the help of the computer is the Scilab software, the free and open source equivalent of Matlab. We comment shortly on an exercise in calculus – numerical integration – proposed to high school students with no school background in the subject.

In our approach, we introduce integration *before* introducing derivation, and the definite integral before the indefinite one. This is in accordance to how these notions have appeared in the history of mathematics. Also, this is the way in which the visual idea – of area – comes before any analytical formula or idea. Then, the derivative is introduced also in relation to the graphical object which is the tangent line. And the Fundamental theorem of calculus becomes the unexpected and striking link between these two visual ideas.

This theoretical introduction, different from any program taught at school that we may know, can be perfectly illustrated and trained with the use of the computer visualizing techniques. In the exercise shown in Fig. 3, the student, who have learned before a few graphical commands, is asked to draw the graphic of the sine function on the segment $[0, 6\pi]$ and of the function $\min(\sin(x), 0.5)$ – which is explained in words and not by a formula in the exercise statement. Then he/she should find a way to approximate the cut area. To this end, an additional tool is given: a Scilab function which reads the coordinates of the point that we click on.

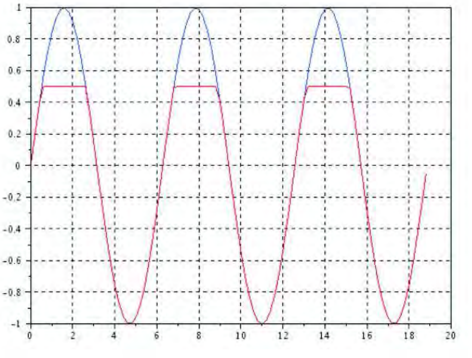
Still, the student can chose his/her own way to solve the problem. The exercise is easy, but demands some creativity and decision taking. It also allows to compare solutions of different students.

Let us note that Scilab has been introduced in French high schools (Gomez et al. 2013), which have one of the most ambitious, detailed and rigorous programs for teachng mathematics in Europe. Scilab gained in this way a special "lycée" module and has been introduced in 2006-2009 at the final school examination (baccalauréat) in classes specialized in mathematics or technology. At this moment, numerical exercises are still present but in a "paper" form: the student is asked to design an algorithm which should complete some task: this supposes some former experience with programming. Scilab stays an excellent software for such an experience.

5. Narysować wykres funkcji *y=sin x* w przedziale [0, 6π].

a) Wyznaczyć przybliżoną pochodną i narysować jej wykres.

b) Narysować wykres funkcji powstałej poprzez zastąpienie wszystkich wartości większych niż 0.5 wartością 0.5 (patrz rysunek).



c) Narysować wykres funkcji powstałej poprzez zastąpienie wszystkich wartości większych niż 0.5 wartością 0.5 oraz mniejszych niż -0.5 wartością -0.5.

d) Jak w sposób przybliżony można obliczyć pole 'obciętej' części wykresu? Proponowane narzędzia: całkowanie numeryczne, może funkcja `[buttons,x,y]=xclick()....`

Figure 3. An exercise with Scilab – learning numerical integration.

3. Experimenting

Many mathematical discoveries were fruits of a long and tedious process, in which mental or drawing experiments played a key role, much more important than what we can suspect when learning. The godfather of all contemporary signal processing and electronics, Jean-Baptiste Joseph Fourier (1768-1830), left a great number of graphics representing sums of scaled sine functions (Fourier, 1822). They give as an impressive insight into a fertile mental process. Before formulating his crucial theorem on the development of a function into a series of sine functions, he was just "playing" with drawing a sum of two, three, five of these and observing the result. Isn't it just experiment? His observations were certainly more valuable than the exactness of the resulting statement – the formulation of the latter had to be corrected by Fourier's successors.

Do we leave any place for such a mental process to our students? Let us give a few ideas of how to introduce experimenting into the classroom.

Some examples of classical experiments; exercises as experiments

The most adapted areas for experimentation are certainly probability and geometry. In probability, some of these are described in polish handbooks (see Karpiński et al., 2011), like the Monty Hall game. Other experiments are worth to be recommended, like the Galton box or the "liar" test: take two students, give them two sheets of paper, two pencils and one coin. Ask them to leave the classroom and note fifty results of dashes. One of them shall note real results, the other shall invent them. The teacher, who does not know the roles, will in most cases be able to guess which results are real. This is the result of the "series law", which is quite counter-intuitive: in a series of fifty dashes the probability of getting a series of five consecutive shoots is close to one. And it will nearly never appear when the results are invented by a human, the verification of which we leave to the reader as well!

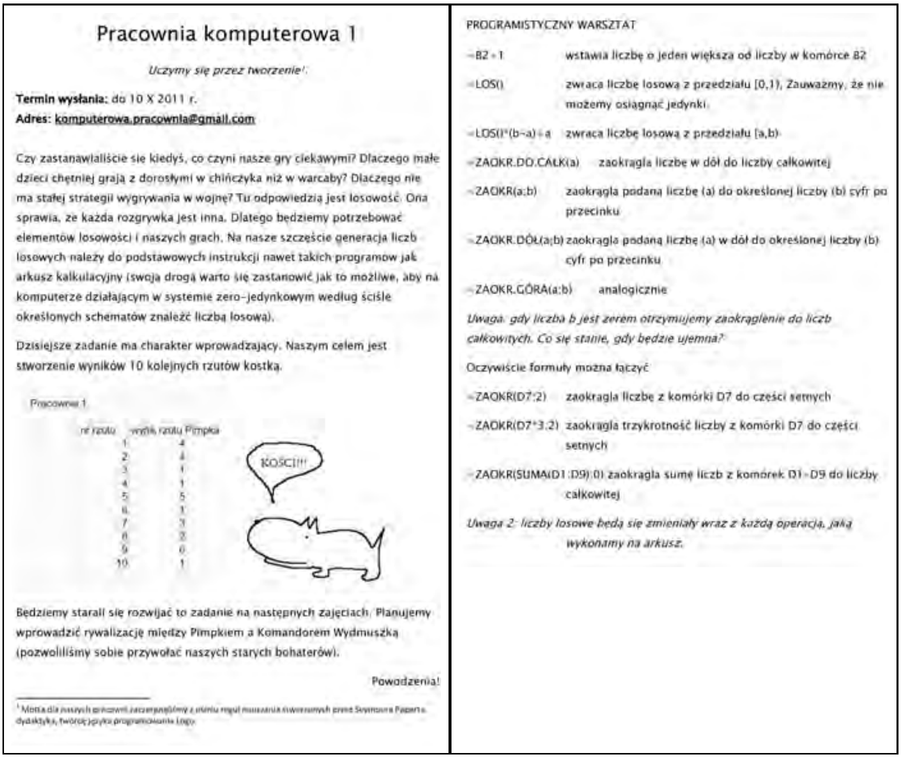
Let us still retain your attention on a mental experiment – counter experiment? – which are the graphics of the Dutch artist M. C. Escher (see the official webpage www.mcescher.com). Many of them deal with the non-Euclidean geometries (Circle Limits, Positive/Negative Space, Snakes, Herakleidon etc). They can also be viewed as a visualization for the mathematical notion of infinity, infinite sequence or series. Finally, the large number of graphics which may be the most interesting from the point of view of a teacher represent paradoxes and optical illusions (see e.g. Relativity, Waterfall, Stairway). They captive the pupil's attention and interest, but also make the Euclidean axiomatic and geometric proofs meaningful.

An experiment is a test of a new hypothesis, an observation, a contact with the real, tangible world. But experimenting means also repeating the same action under varying conditions. This has always been present in teaching mathematics in the form of exercises. It seems to be somewhat neglected nowadays in polish school handbooks: they contain much less exercises than those used twenty years ago. We want to stress that exercising does not have to be just memorizing an algorithm. It is does not have to go against thinking. Indeed, it can also be experimenting, and we are deeply convinced it should be seen in this way by teachers. This may give sense to any homework, especially the one which is not only repetition, but repetition in a new situation. This also gives sense to the idea of a “flipped classroom”, when homework is a preparation for the lesson on the subject, and the student’s work is to try to individually cope with a new situation, a new area, which is then commented in the class – see the book of Bergmann and Sams (2012) and many other recent publications on the subject available on the web. It finally gives much sense to projects realized by students in relation with the learned subject. The GWO Polish handbooks (Karpiński et al., 2012; Karpiński et al., 2011) give now a beautiful collection of propositions in this direction.

Examples with use of the computer

The three examples of visualization given in Section 2. introduce clearly the possibility of experimenting, or simply playing – with the tool, which is the computer, and with the mathematical language. Can you change the system of equations in Example 1 so that the two lines do not intersect? What does it mean for the solution? What would give a different combination of sines in Example 2? Do you think there is some “limit” function? What would it be? How does the value of the area you calculate changes when you take thinner rectangles, or trapezes? These questions give place to deep problems and may introduce practical understanding of abstract notions: existence and uniqueness of solution, limits, convergence – without any abstract definition yet. The students can then try to give their own definition that we may discuss. In this way, we really do mathematics, discovering it as an art of ideas, as postulated in the beautiful essay of Lockhart (2009). Learning mathematics becomes again a “creative and rewarding process of invention and discovery” (Lockhart, 2009).

Let us give still another example, this time in probablity. In the exercise presented in Fig. 4, the students are asked to simulate a dice on a spreadsheet (such as Excel, Open Calc, Google Docs). All the commands that one may need to do this are given. The student has to use the given command producing a random number between 0 and 1, and then transform the result into a random integer between 1 and 6 by one among many given methods. Besides choosing a way to do so, he or she will have to realize that the right solution has to really simulate a symmetric dice,



Pracownia komputerowa 1

Uczymy się przez tworzenie[1].

Termin wysłania: do 10 X 2011 r.
Adres: komputerowa.pracownia@gmail.com

Czy zastanawialiście sie kiedyś, co czyni nasze gry ciekawymi? Dlaczego małe dzieci chętniej grają z dorosłymi w chińczyka niż w warcaby? Dlaczego nie ma stałej strategii wygrywania w wojnę? Tu odpowiedzią jest losowośc. Ona sprawia, ze każda rozgrywka jest inna. Dlatego będziemy potrzebować elementów losowości i naszych grach. Na nasze szczęście generacja liczb losowych należy do podstawowych instrukcji nawet takich programow jak arkusz kalkulacyjny (swoją drogą warto się zastanowić jak to możliwe, aby na komputerze działającym w systemie zero-jedynkowym według ściśle określonych schematów znaleźć liczbą losową).

Dzisiejsze zadanie ma charakter wprowadzający. Naszym celem jest stworzenie wyników 10 kolejnych rzutów kostką.

Pracownia 1

nr rzutu	wynik rzutu Pimpka
1	4
2	4
3	1
4	1
5	5
6	1
7	3
8	2
9	6
10	1

Będziemy starali się rozwijać to zadanie na następnych zajęciach. Planujemy wprowadzić rywalizację między Pimpkiem a Komandorem Wydmuszką (pozwoliliśmy sobie przywołać naszych starych bohaterów).

Powodzenia!

[1] [illegible]

PROGRAMISTYCZNY WARSZTAT

=B2+1 wstawia liczbę o jeden większa od liczby w komórce B2

=LOS() zwraca liczbę losową z przedziału [0,1), Zauważmy, że nie możemy osiągnąć jedynki

=LOS()*(b−a)+a zwraca liczbę losową z przedziału [a,b)

=ZAOKR.DO.CAŁK(a) zaokrągla liczbe w dół do liczby całkowitej

=ZAOKR(a;b) zaokrągla podaną liczbę (a) do określonej liczby (b) cyfr po przecinku

=ZAOKR.DÓŁ(a;b) zaokrągla podaną liczbę (a) w dół do określonej liczby (b) cyfr po przecinku

=ZAOKR.GÓRA(a;b) analogicznie

Uwaga: gdy liczba b jest zerem otrzymujemy zaokrąglenie do liczb całkowitych. Co się stanie, gdy będzie ujemna?

Oczywiście formuły można łączyć

=ZAOKR(D7;2) zaokragla liczbę z komórki D7 do części setnych

=ZAOKR(D7*3;2) zaokrągla trzykrotność liczby z komórki D7 do części setnych

=ZAOKR(SUMA(D1:D9);0) zaokragla sumę liczb z komórek D1–D9 do liczby całkowitej

Uwaga 2: liczby losowe będą się zmieniały wraz z każdą operacją, jaką wykonamy na arkusz.

Figure 4. A first homework with a spreadsheet (an exercise sheet in Polish).

i.e., to give all the six results with the same frequency. The student has then to experiment with what he/she has created. Simulating one thousand shoots presents no more difficulty here than simulating just one. The only problem may possibly be to count the frequency of each of the numbers, obtained and to realize what is the mistake I have done, if this frequency does not seem equal for all the six numbers. We gave this task to a group of 20 high school students (of the course mentioned in the Introduction) and to about 40 mathematics teachers of mathematics during training courses at the Cardinal Stefan Wyszyński University in Warsaw. About 80% of participants in both groups chose a method which produced this non–symmetric effect, among the others some avoid it just by chance, and a few are able to imagine the result already at the command construction level. In other words, this experiment becomes again a real discovery leading to a deeper understanding. Besides, the semi-random numbers are an interesting subject to be explained at this occasion.

This exercise is only a first of seven steps, leading finally to creating a game. Each step is composed of a description of the problem to be solved, of some comments, a collection of useful tools – commands and finally a sample of a simple story with a naive illustration. At the same time that the students are creating the game, they also learn the programming structures such as the if/else statements, loops etc. They can go above what is necessary and use more commands to introduce more advanced functions or graphics.

Our experience is that, as well for the pupils as for the teachers, the possibility of simulating a random process is a new experience and that it is seen as an interesting and powerful tool.

Let us note that these examples enter again perfectly into "flipped classroom" model. Actually, when teaching to the group of high school students, the game creation described above was their almost fully individual work – they were given the exercise sheet with the spreadsheet tools and were asked to send the result by e-mail to the tutor who commented the solutions, possibly showing the mistakes and asking for a new solution. A few classes took place for summing-up the progress. Experimenting should indeed be carefully prepared and commented by the teacher and not only left as a trial-and-error search, which has been seen already by Zhu and Simon (1987, p. 156). At the same time, by experimenting, the student is not confronted to the teacher's decision (right/wrong), but to a situation (works/does not work). The key role of this didactic change was seen and worked on by by Seymour Papert (1980; 1993), the inventor of Logo, in particular in relation to computer tools in learning mathematics. Finally, also following Papert (1983) and Lockhart (2009), let us note that experimenting has many features of what is a play and in this way is not only instructive, but also pleasant for the student.

4. Discusssion and conclusion

We gave a few examples of visualizing and experimenting in mathematics that we strongly believe highly applicable and useful in teaching. The two methods seem to be closely related, especially when applied with the use of the computer.

We are aware of the fact that we enter here the vast research and discussion, which also go back to the 80's and started probably with a critical position of Clark, who argued that "media do not cause learning" (Clark, 1983, p. 457). A number of arguments in the debate which emerged has been extensively cited or summarized in Clark's book (2001). Let us note first a few points situating us in this discussion.

1. We are not looking at computers as vehicles of any dramatic change in education, but as mere instruments for what may be also done by more conventional methods. However, we see a range of technical possibilities in teaching and learning mathematics given by computers.
2. We nowhere argue for any "computer-based instruction", which is one of the key words of the debate that we refer to. On the contrary, whereas looking for effective methods, programs and tools, we strongly believe that instruction, especially instruction of non-adult students, is based on the teacher, on its own relation to the subject that he is teaching and to the students. In this context, we note that our "computer-based" classes (i.e. those in a computer lab) were in general taught by teachers who were willing to give them and felt competent in the subject. However, they did not have to be specialists neither in computers nor in the software they were using.
3. The same applies to students, who were in general free to chose the computer classes. About one third of our students were not interested with experimenting with computers by themselves. At the same time, visualizations done with the help of the computer, were rather always interesting to everybody.

We fully agree with Watson's (2001) conclusion: "We need to inrtervene with educational ideas, not simply technological issues". Teachers do not have to master the techniques, but to be convinced by the pedagogical ideas behind. Acting in this spirit, we tried to present some of them, and we have done it following the methodology of "learning from examples and by doing" (Zhu & Simon, 1987), or what we call experimenting. Our experience leads us to the opinion that mastering a technique needed for a particular example, if coming from the conviction about its didactical utility, is enough for the teacher to use it in the way to put more mathematics and more pedagogy in the class. As we have shown in Section 2., the technical part can be minimal.

If we avoid the mistake of giving some determining power to the technology, we believe we may also prevent other risks. Among them those traditionally attributed to the use of computer tools in education: on one hand, the overdose of visual or acoustic impulses which can affect the ability to concentrate, certainly essential in learning mathematics, and on the other the absence of the abstract and formal setting, which, even if less essential, is also necessary in teaching and learning, and can be essential to some of the students (Clements, 2014 and references therein). When the computer in education is thought as a tool for visualization and experiment, and not as a source of information or a technique to master; when visualization and experiment are seen as methods of filling some gap in the education, and not a goal in themselves, finally when the teacher keeps its crucial role in the teaching process, we believe that these risks are very much attenuated.

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The use of Stanisław Dróżdż's works as teaching aids in mathematics

Abstract. As far as teaching practice in mathematics is concerned, we use different methods, forms and tools, which enable pupils to acquire both theoretical knowledge and practical skills more efficiently. Stanisław Dróżdż (1939-2007) was a concrete poet, who, in his work, used not only words, but also visual art. Poetry, art and mathematics are apparently distant domains, yet when I saw Dróżdż's works, I noticed the opportunity to use them as original tools in teaching mathematics. I will show his chosen works and different ways of using them in mathematical education. Some of them could be used as illustrations of various concepts (for example at early stages of maths education). Others could be starting points in discovering and exploring some regularities. There are also some which could inspire pupils to look into certain mathematical problems.

Whilst creating his concept-shapes, Stanisław Dróżdż used similarities and differences of various situations. Searching for similarities and differences using analogies and contrast also makes it possible to build mathematical concepts in pupils' minds. By putting together concept and shape, Dróżdż worked in a way similar to practice in mathematical education – while defining concepts, we give them a name and (very often) a symbol, and we try to visualize them by giving them a "shape". Using Stanisław Dróżdż's works, it is possible to build yet another bridge between theory and practice, particularly for pupils to whom maths is not a life-long passion.

A teaching aid is "a tangible object which facilitates the process of teaching-learning and enables pupils to achieve school optimum" (Okoń, 2007), or "an object that provides students with specific sensory stimuli

Key words and phrases: theory, practice, teaching aids, working methods.
AMS (2000) Subject Classification: 97A30.

that affect their sight, hearing, touch, etc., facilitating their direct and indirect cognitive processes" (Kupisiewicz, 2009). Both definitions refer to objects that are used so as to enhance the acquisition of new knowledge or skills, or deepen the understanding of those already existing. In teaching practice, it is impossible to separate the object from the function it is to perform. The following functions of teaching aids are distinguished: 1) exemplifying concepts, 2) facilitating thought processes, 3) assisting pupils in exercises aimed at acquiring practical skills, 4) inspiring students' emotional reactions (Okoń, 2007). Currently, there is a wide range of teaching aids on offer in each of these categories (verbal, visual, technical-visual, auditory and visual-auditory, as well as those which automate the process of teaching) (Okoń, 2007). However, we should not overlook any opportunity to make use of our own creativity when encountering objects which may be reasonably used as teaching aids. Such was the case of works by Stanislaw Dróżdż that I had the opportunity to come across. I will present some, perceived by me, uses of his selected works in teaching mathematics.

The first of the above-mentioned functions of teaching aids is exemplifying concepts in the learning process.

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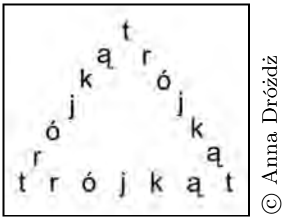
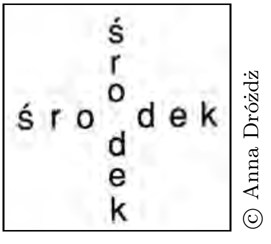
Work 1. *między* (za szybami)/*between* (behind the glazings), 1994, Muzeum Współczesne/Contemporary Museum, Wrocław

Pupils in primary education tend to have problems with describing relationships between objects. Demonstrating and discussing concept-shapes created by Stanislaw Dróżdż, e.g. *między* (za szybami)*between* (behind the glazings), 1994 (work 1) with children, can become a starting point for pupils' autonomous, creative activities, so that following the poet's idea they could try to create "self-realising" (Łubowicz, 2014) objects. At the exhibition showing the works by Stanislaw Dróżdż in Wroclaw Contemporary Museum, you can see more works of this type.

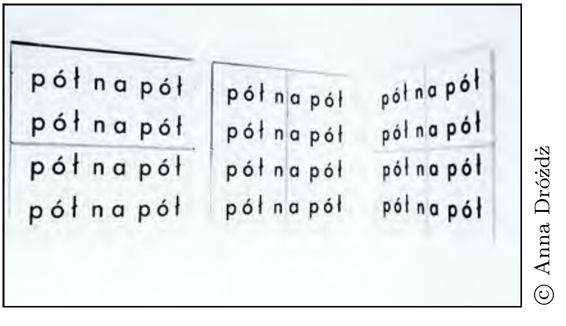


Work 2. Bez tytułu (trójkąt)/untitled (triangle), 2006, http://www.drozdz.art.pl

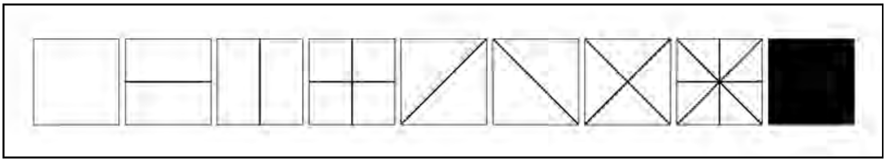


Work 3. Bez tytułu (środek)/untitled (center), 1998, http://www.drozdz.art.pl

Students on higher educational levels may try to use the idea of describing an object's shape, using its name, both in literal – as shown in the work *bez tytułu* (trójkąt) *untitled* (triangle), 2006 (work 2) and an abstract way – like in *bez tytułu* (środek)/*untitled* (center), 1998 (work 3), where the practically described object was not "drawn", although we have no doubt as to its existence.



Work 4. Bez tytułu (półna pół)/untitled (half by half), 1998, Galeria Potocka/Potocka Gallery, http://www.drozdz.art.pl

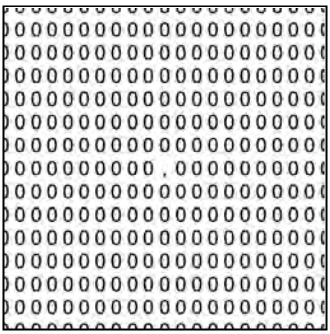


Work 5. *Sytuacja semiotyczna/Semiotic situation*, 2006, http://www.drozdz.art.pl

You can also attempt to present certain terms. In his work *bez tytułu* (półna pół)/*untitled* (half by half), 1998, (work 4) the poet showed how a whole ("half by half" written four times) was first divided in half in two different ways, and how "half by half" (i.e. a quarter) was worked out. A similar idea (though not only) can be seen in part of the work

Sytuacja semiotyczna/Semiotic situation, 2006 (work 5). The second and the third square (counting from the left) shown in this work, are divided into half segments contained in the square symmetry axis passing through the centers of the respective sides, and the fourth shows the imposition of these two situations. Similarly,the fifth and the sixth squares are cut into halves by their diagonals, and the seventh one shows alignment of these two situations.

With this work we can go forward, because the eighth square can be treated as imposition of the situation with the fourth and the seventh squares (and we can ask students to interpret what we obtain in this manner). We may be tempted to interpret the first and the last square in this sequence, but that is another problem.



Work 6. *Continuum*, 1973, http://www.drozdz.art.pl

In the process of learning we often encounter situations in which students come up against difficulties, not only in understanding concepts themselves, but also in using them correctly. When learning decimals, very often pupils do not know which zero can be omitted without changing the number. *Continuum*, 1973 (work 6) is well-suited as a starting point to talk about it. The title suggests that we are dealing with something that lasts. And indeed, presented there, the "badly cropped“ (truncated) image suggests that we are dealing with only a fragment of a larger whole, which ... extends to infinity. Basing on experience (when a large number is saved, whose record cannot be fitted in one line, it is continued in the next line), we can assume that the present record shows a number. Assuming that the same characters (i.e. 0) “extend” indefinitely in all directions, then, although it occupies the whole plane, the value shown here is the number of ... 0. Interpreting this work performs the fourth function to be fulfilled by teaching aids – inspiring students' experience. Talking about what would happen if e.g. *not all characters were zeros before the decimal point, or not all the characters were zeros after the decimal point*, rounds up the substantive content of the issue in question.

Work 7. Bez tytułu (punkt)/untitled (point), 2006, http://www.drozdz.art.pl

Contrasting statements are very often used in teaching. In the work *bez tytułu* (punkt)/*untitled* (point), 2006 (work 7) on the one hand, we have to deal with such contrasting juxtaposition, but we can examine it further. If we ask questions, eg.: *Are there really no points on the left? Is there only one point on the right? Is it only a distinction? Why is it so big?*, this work can serve as a starting point to talk about understanding the object in question – the point. Another kind of contrast was used in the work *bez tytułu* (koło)/*untitled* (circle), 1971 (work 8). For my own private use, I call this work “rectangling the circle”, although the association is only visual and the problem of squaring the circle has nothing to do with it. The presented shape of a rectangle is puzzlingly juxtaposed with the word “circle”, while the eye of a beholder reading the word is moving in a circle; this could be a beginning of a conversation on the nature of the object in question – the circle.

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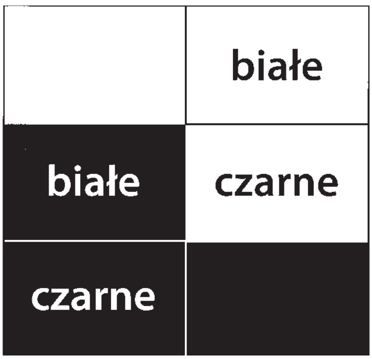
Work 8. Bez tytułu (koło)/untitled (circle), 1971, Galeria Foksal/Gallery Foksal, Warszawa (Stanisław Dróżdż *Pomysły*)

The second function of teaching aids is facilitating thought processes.

Three works by Stanislaw Dróżdż, presented here, can initiate a process of thought in different areas of mathematics, and at the same time, through their very structure, they can help pupils to carry out reasoning.



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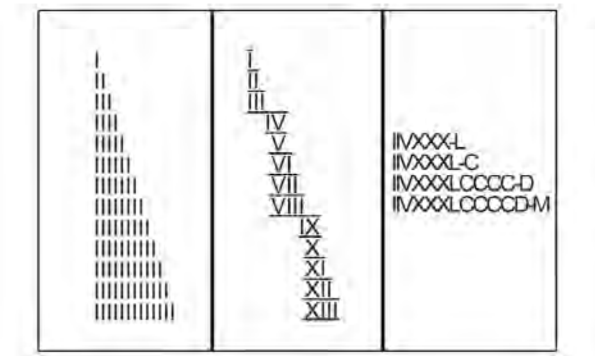
Work 9. Bez tytułu (białe – czarne)/untitled (white – black), 1970, http://www. drozdz.art.pl

The first of these is the work *bez tytułu* (białe – czarne)/*untitled* (white – black), 1970 (work 9), followed by some questions: *Is there any message in the top left hand corner and the bottom right hand corner?If so, what is it, and why? If not, why not?*



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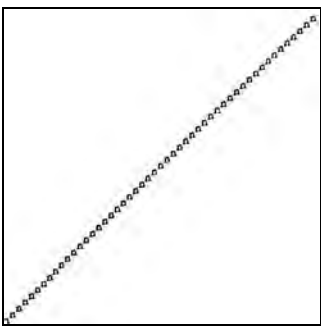
Work 10. Bez tytułu (cyfry rzymskie)/untitled (Roman numerals), 2006, http:// www.drozdz.art.pl

The second one is the work *bez tytułu* (cyfry rzymskie)/*untitled* (Roman numerals), 2006 (work 10) with questions: *Are the three parts of this work somehow related to each other? If so, how? If not, why not?*



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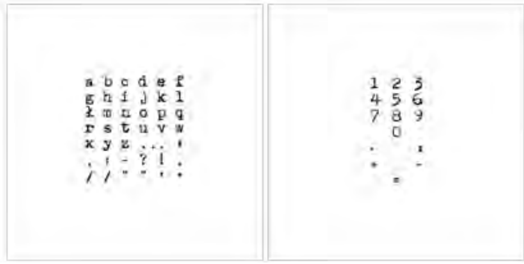
Work 11. Bez tytułu (potęgowanie)/untitled (exponentiation), 1981

The third work is *bez tytułu* (potęgowanie)/*untitled* (exponentiation), 1981 (work 11), and the questions are: *What sort of exponentiation is being dealt with? Why*?

Each situation presented here allows you to touch certain mathematical problems.



Work 12. *Język i matematyka/Language and Mathematics*, 1968, http://www. drozdz.art.pl

To express our thoughts we use symbols. The last work that I want to present in this section is *Język i matematyka /Language and Mathematics*, 1968 (work 12). The author put together the symbols used in language and mathematics. Examining this juxtaposition, as well as each part separately, may make students consider some questions, e.g.: *Do all the signs listed here have the same functions? If so, what are they? If not, how do they differ? Are those all the symbols used in a given field (as far as you know)? If not, which ones were left out and why? Are there any symbols that occur in both parts? If so, do they mean the same?* After this analysis (which will proceed differently at each level of education), the symbols used and the contexts of their use will become clearer for students. The conversation related to this work can also be seen as a starting point to explore the history of presented objects.

The third function of teaching aids is assisting pupils in exercises aimed at acquiring practical skills. "Practical skills" can be understood in several ways. Among others, it might mean practical use of mathematics in everyday life (whether private or professional); it could be perception of mathematics in our surroundings (in the works of nature or creations of human hands), and it can also be physical activity which helps to investigate a mathematical problem.

In this section I will present some works by Stanislaw Dróżdż that use mathematics so that firstly, they present some task, the analysis of which requires a selection of a suitable mathematical model for a given situation.

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Work 13. *Klepsydra* (było, jest, będzie)/*The Hourglass* (it was, it is, it will be), 1967, Muzeum Narodowe/National Museum Wrocław

The work *Klepsydra* (było, jest, będzie)/*The Hourglass* (there was, there is, there will be), 1967 (work 13) is exhibited in the National Museum in Wroclaw. Interpreting this work performs the fourth function to be fulfilled by teaching aids – inspiring students' emotional experiences. Very simply and aptly, the author shows how insignificant is the part of reality in which we live, and at the same time how unique it is. The mathematical task is connected with the poet's successive approaches to this work.

In Wroclaw Contemporary Museum the work can be seen in extensive form (work 14). With a question, *Why does the presented series consist of 54 boards?,* it poses a very interesting combinatorial problem. A practical approach to the theoretical mathematical situation is realized here through the use of a specific artistic work as a starting point, which allows us to apply and justify the use of an appropriate theoretical model – *In order to define such work, did the artist use permutations, variations or combinations, with or without repetition?*

In this regard, we can use some other works by Stanislaw Dróżdż like "*i*" *(fragmenty)/"and" (fragments)*, 1970-1997 (work 15) and *bez tytułu* (równa się, nie równa się)/*untitled* (equal to, not equal to),1971-1972 (work 16), but the culmination is the work *Alea iacta est*, 2003 (work 17a) with which the author represented Poland at the 50th Venice Biennale. The latter work can be used like discussed earlier *Hourglass,* giving instructions (work 17b), which the author posted on display in many languages, but you can also(without giving instructions) ask: *How many dice* *are there lining the walls of this pavillion,if you can find there all possible outcomes of throwing six classical dice*? Considering different rules that you can use “lining the walls", there may be more than one answer.

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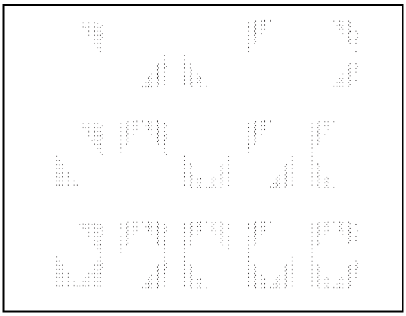
Work 14. *Klepsydra* (było, jest, będzie)/*The Hourglass* (it was, it is, it will be), 1967-1990, shows a fragment of work consisting of 54 boards, Muzeum Współczesne/Contemporary Museum, Wrocław



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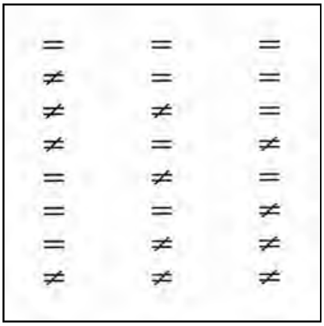
Work 15. *“i” (fragmenty)/“and” (fragments)*, 1970-1997, http://www.drozdz.art.pl



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Work 16. Bez tytułu (równa się, nie równa się)/untitled (equal to, not equal to), 1971-1972, http://www.drozdz.art.pl

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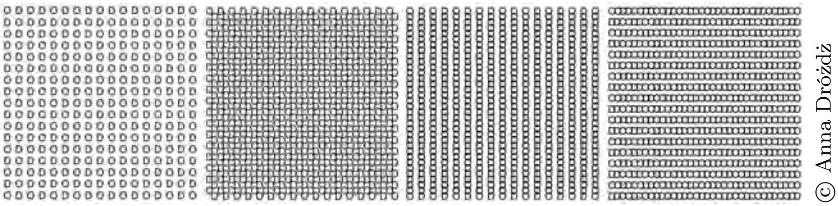
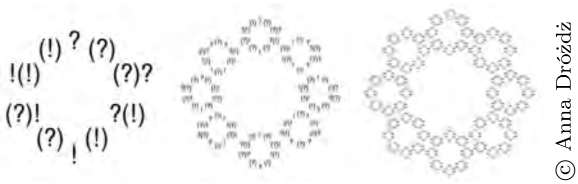
Work 17a. *Alea iacta est*, 2003, 50. Biennale w Wenecji, Pawilon Polski, http://www.drozdz.art.pl

a game of dice
rules of the game by Stanisław Dróżdż:
there are 46,656 possible results of casting 6 dice
all these configurations have been placed on the wall.
take the dice lying on the table and cast them
align them in one row
try to find exactly the same combination on the wall
If you find it, you win, if not, you lose,
and it's me who wins.

Work 17b. Instrukcja – część pracy *Alea iacta est*, 2003, 50. Biennale w Wenecji, Pawilon Polski, http://www.drozdz.art.pl



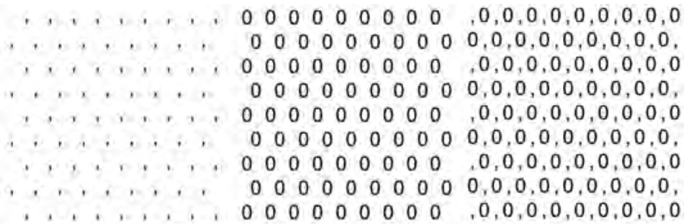
Work 18. *Czasoprzestrzennie* (OD – DO)/*Temporally – Spatially* (FROM – TO), 1969–1993, fragment of work consisting of 82 boards, http://www.drozdz.art.pl



Work 19. Bez tytułu (wykrzykniki i znaki zapytania)/untitled (exclamation, question marks), 2001, fragment of work consisting of 13 boards, http://www.drozdz.art.pl

The work *Czasoprzestrzennie* (OD – DO)/*Temporally – Spatially* (FROM – TO), 1969–1993 (work 18) and *bez tytułu* (wykrzykniki i znaki zapytania)/*untitled* (exclamation, question marks), 2001 (work 19) can be used to locate and describe geometric transformations we would deal with if we wanted to turn one work into another. Here, depending on the age and abilities of students, we can have a practical approach to a theoretical mathematical situation reduced to finding a suitable transformation and describing it with known symbols, or, alternatively, equipping students with appropriate copies, eg. on transparencies, physically allowing them to find appropriate transformations and try to describe them.

Work 20. *Tryptyk* (Niepewność – Wahanie – Pewność)/*Triptych* (Uncertainty – Hesitation – Certainty), 1967, http://www.drozdz.art.pl



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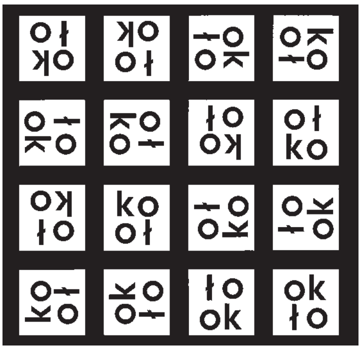
Work 21. Bez tytułu (przecinki, zera)/untitled (commas, zeros), 2006, http://www.drozdz.art.pl



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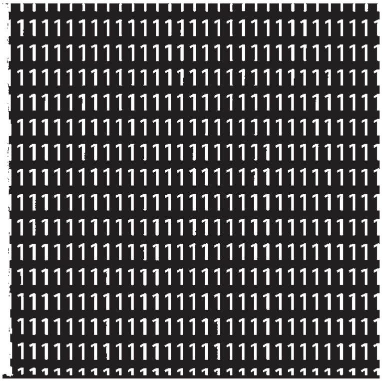
Work 22. Bez tytułu (koło)/untitled (circle), 1971-1972, Stanisław Dróżdż. *Początekoniec. Pojęciokształty. Poezja konkretna Prace z lat 1967-2007*

For this purpose, we can also use other works by the poet, for example: *Tryptyk* (Niepewność – Wahanie – Pewność)/*Triptych* (Uncertainty – Hesitation – Certainty), 1967 (work 20), *bez tytułu* (przecinki, zera)/*untitled* (commas, zeros), 2006 (work 21) or *bez tytułu* (koło)/*untitled* (circle), 1971-1972 (work 22). One version of the latter work is to be found in the public space in Wroclaw, so discussion can be carried out while physically facing the work, outdoors.

The fourth function of teaching aids is displaying materials that inspire students’ emotional experience.

The implementation of this function with the use of works by Stanisław Dróżdż has already been mentioned while discussing *Continuum*, 1973 (work 6) and *Klepsydra* (było, jest, będzie) /*The Hourglass* (there was, there is, there will be), 1967 (work 13). It can also refer to the work *Tryptyk* (Niepewność – Wahanie – Pewność)/*Triptych* (Uncertainty – Hesitation – Certainty),1967 (work 20), but there are two other pieces, in which I see the essence of mathematical objects used to describe life situations.



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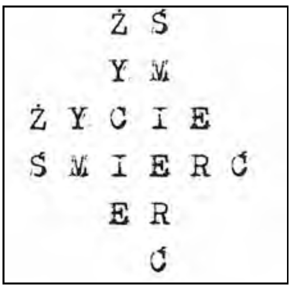
Work 23. *Samotność/Loneliness*, 1967, http://www.drozdz.art.pl

The first work is *Samotność/Loneliness*, 1967 (work 23). It shows number 1 multiplied, which on the one hand represents a whole, but on the other hand also means individuality. Replicated with a clear sense of space, it becomes a work of art – a wonderful interpretation of the title *Loneliness* (alienation, lack of communication).



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Work 24. Bez tytułu (życie – śmierć)/untitled (life – death), 1969, http://www. drozdz.art.pl

The second work is *bez tytułu* (życie – śmierć)/*untitled* (life – death), 1970 (work 24). The terms describing both conditions begin simultaneously – which is indicated by the simultaneity of the existence of both. But both words are written in such a way that they do not overlap – in two parallel lines – which in turn indicates their separation and independence.

Additionally, the author wrote these conditions using two different directions – vertical and horizontal. Thus, right angles have been suggested, which may symbolize intersecting fates of different lives, but in a very "orderly" way (the only perpendicular lines divide the plane into four equal parts). At the same time, the perspective from which we view this work is changing what we see (which is not without significance). When we look at it very closely (in one's youth), we are able to see only selected parts. With increasing distance (elapsed time), we better perceive the whole, we see the relationships. From a sufficiently large distance, we see only the outline of two straight perpendicular lines.

All these interpretations shown here and uses of works by the poet Stanislaw Dróżdż are mine and are not unique, but the presented examples prove that different objects might become teaching aids. For those visiting Wrocław, it might be worth knowing that two outdoor projects can be seen while walking around the city: one of the variants is *Hourglass* – on the building of Wrocław Contemporary Museum and *untitled (circle)* – in Nowy Targ square, in front of the Municipal Office.

I offer heartfelt thanks to Ms Anna Dróżdż for providing the photos of the works by Stanisław Dróżdż and contact with the people involved in his work; to Ms Ewa Trojanowska for providing valuable guidance related to the description of the works and the history of their formation.

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Circle and sphere – geometrical speculations in philosophy

Abstract. The circle and the sphere, in philosophical speculations, exist as symbols of perfection, as metaphors of divinity, as models of eternity as well as approximations of essential properties of cognitive acts. Their geometry is also an excuse for visual speculations of an artistic nature. In this article, we discuss some chosen metaphors based on the circle and sphere which refer to both ontological and epistemological issues pertaining to various models of knowledge and the cognitive process.

1. Introduction [1]

The short essay by Jorge Luis Borges "The Fearful Sphere of Pascal" starts out with words which could stand in as the motto of this article: "It may be that universal history is the history of a handful of metaphors" (Borges, 1964, p. 168). Borges describes the metaphors based on a sphere which appear in history by referring to among others Empedocles', Giordano Bruno's or Pascal's speculations. This essay focuses around the famous metaphor quoted after Alain de Lille, in which "God

Key words and phrases: circle and sphere in philosophy and art, visual metaphors in ontology and epistemology, Nicolas of Cusa.

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1. The authors are grateful to the anonymous reviewer's valuable comments that improved the manuscript.

is an intelligible sphere whose centre is everywhere and whose circumference is nowhere" (Borges, 1964, p. 169)[2]. This article is also a review of other philosophical metaphors based on roundness pertaining not only to the sphere but also to the circle. In this instance we refer to the thoughts of Parmenides, Plato, Nicolas of Cusa, Quine, Heller and Twardowski. Metaphor is understood here broadly in the sense proposed by Lakoff and Johnson (1980) as a certain model representing chosen aspects of a given issue in terms characteristic for a different issue.

The circle and the sphere, in philosophical speculations, serve as symbols of perfection, as metaphors of divinity, as models of eternity as well as approximations of properties of cognitive acts. Their geometry may also be an excuse for artistic visual speculations. In our proposed approach, art, hand in hand with mathematics and philosophy becomes a cognitive tool – it constitutes not only an aesthetic complementation of scientific cognition, but it is its reinforcement, deepening and extension.

The article consists of two parts. In the first, we discuss metaphors which appear in ontological problems. Ontology is treated here as the most general science on what there is and what may be, and speaking in the language of traditional philosophy: as the science of existence or being. Within such a wide understanding of ontology we are allowed to also take up metaphysical issues which are related to real existence and not only possible existence[3]. In the second part of the article we discuss epistemological traits pertaining to various models of knowledge and cognitive process.

In this article we mainly focus on metaphors which are based on the analysis of the geometrical properties of the sphere and the circle and not only on the superficial properties of these figures visible to the naked eye. This harmonious fusion of geometrical analysis, philosophical speculation and artistic visualization offers an opportunity for wider comprehension and an intelligible approximation of the studied subject.

2. Original: "Deus est spaera intelligibilis, cuius centrum ubique, circumferentia nusquam" (Alanus De Insulis, VII PL 210, 0627A).

3. On the relation between metaphysics and ontology see (Perzanowski, 1988, pp. 87–90).

2. Circle and sphere in ontological metaphors

2.1. Spherical vision of reality by Parmenides

Parmenides was supposedly the author of one treaty of which only a few fragments have been preserved[4]. However his thought has played an important part not only in Greek philosophy but also in later philosophies. In the Prologue to his poem Parmenides recalls the motif of a journey; a journey from the land of night to the land of day and light. That journey symbolizes cognition which starts with sensual experience but does not end with just that. As Parmenides wrote, the presumptions of mortals are based on senses, but the truth may not come from presumptions alone. Therefore, true cognition needs reason to act as a guide. A goddess welcoming travelers says (Kirk and Raven, 1957, p. 267):

> It is no ill chance, but right and justice, that has sent thee forth to travel on this way. Far indeed does it lie from the beaten track of men. Meet it is that thou shouldst learn all things, as well the unshaken heart of well-rounded truth, as the opinions of mortals in which is no true belief at all.

As early as in the prologue the motif of roundness appears. Reason leads to true cognition, to a Truth which is rounded and similar to a sphere.

Parmenides following the path of light and reason, described existence [Parmenides called it "true reality"] in many aspects. Most of all true reality is what is and not what may be. It has not been born and is indestructible, it has never been created and it cannot die. It is a wholeness which is filled, in every part, in the same way. It may not be more in one place and less in the other, there is the same amount of true reality in every place. True reality is continuous and undividable. It is also unimpressed and unchangeable. Such a characterization of true reality leads to regarding it as perfect. In the rationale of that perfection Parmenides writes: "But since there is a furthest limit, it is bounded on every side, like the bulk of a well-rounded sphere, from the centre equally balanced in every direction" (Kirk and Raven, 1957, p. 276). Therefore true reality resembles a sphere, it is sphere-like. The interpretation of that fragment is not a trouble-free task; it is especially hard to establish how Parmenides understood the limit. This fragment may be understood, for instance, that there is the furthest limit of true reality which is the sphere which surrounds it, while true reality is similar to an open ball i.e. the space inside a sphere (without the boundary points).

4. More on the life and work of Parmenides may be found in (Kirk and Raven, 1957, pp. 263–285).

In the contemplations of Parmenides, although they are not unambiguous, the motif of roundness as one of the attributes of perfection has come up. True reality is perfect; therefore it is similar to a sphere. What testifies to that perfection? True reality is "(...) equal to itself on every side, it rests uniformly within its limits" (Kirk and Raven, 1957, p. 276).

2.2. Plato's roundness of the world

The idea of perfection by Parmenides was taken up and developed by Plato. In the dialog *Timajos* he describes the creation of the world. The organizer of the world's creation, the world's craftsman is the platonic Demiurg (Greek: *demiurgos*). Demiurg using the eternal and perfect models has created the world out of fire, air, water and earth. The world's perfection which is the reflection of the form's perfection is demonstrated in the spherical shape the world was given (Plato, 33 BC):

> Wherefore he made the world in the form of a globe, round as from a lathe, having its extremes in every direction equidistant from the centre, the most perfect and the most like itself of all figures; for he considered that the like is infinitely fairer than the unlike.

Amongst other properties, what proves the sphere's perfection is the smoothness of its surface. That smoothness for Plato symbolized self-sufficiency which was understood by him as the absence of any needs. Although Plato understood the world as a living creature, he stated that no limbs protrude from it (legs or arms) which might serve as means to grasp onto subjects or to move. Beyond the world there was nothing that could be held onto and nowhere it could move to. The spherical shape of the world resulted in the fact that it did not need anything else for its existence. Although Plato's idea of the world described here may seem bizarre or even grotesque to a contemporary reader, it carries an interesting ontological reference between the spherical shape and self-sufficiency. If we interpret self-sufficiency as existential self-sufficiency, i.e. that a given object exists independently of the existence of other objects, then surprisingly a sphere would become a symbol of one of the most vital characteristics in contemporary ontology[5].

The fact that according to Plato the universe is spherical, results in the impossibility to distinguish between its "top" and "bottom" because the surface of the world is the same everywhere. Perfection does not differentiate into the right and left side, top and bottom side or front and back side. Even if inside the world there was a solid, it could not be going towards the circumference of the world towards the top or towards the

5. See (Ingarden, 1960, p. 132).

bottom because each edge part of the world seen from the inside is the same and rounded the same way (Plato, 62 DE).

Plato's world was not motionless but it moved in a perfect way, i.e. uniformly along a circle. Plato also probably thought that the world revolved around its own axis. He also mentioned seven planets which move along circles around the earth. Along the circle closest to the earth the moon was moving, whereas the sun in Platonist cosmogony circled around the earth in the second cycle (Plato, 38 CD). The circling of the planets along circles created day and night, month and year – a temporal sequence. The creation of time was therefore determined by among others roundness and a circular movement.

While presenting the creation of the world Plato also described the creation of a human being. The most perfect part of the human body is the head because inside it – according to Plato – the mind is located. The head should therefore be of a perfect shape too. That is why the gods copied the perfect shape and gave the human head the shape of a sphere. The following parts of the human body were created in a way to fulfill the orders given by the most divine and round body part (Plato, XVI D). In the Platonist cosmogony, perfection (but also divinity) of the human mind and the perfection of the world, were met in the metaphor based on roundness.

2.3. Geometrical speculations by Nicolas of Cusa

For Nicolas of Cusa (also called Cusanus) the analysis of geometrical properties of infinite objects became the basis for speculations of a philosophical-theological nature. In the work *On Learned Ignorance* he discusses the relationship between the straight line and a curved line and notices that "(...) the circumference of the maximum circle, which cannot be greater, is minimally curved and therefore maximally straight" (Nicholas of Cusa, 1985, p. 21). Therefore the circumference of an indefinite circle must be the same as the straight line. What Cusanus could only have imagined, today we are easily able to present in the form of a moving picture. Figure 1 presents a shot from an animated loop showing growing circles, the arches of which constantly tend to a straight line.

The animation dependent on time may only present a potential identity of a circle and a straight line. Nicolas of Cusa however, when he wrote about the indefinite circle had actual identity in mind and that is the identity he uses to refer to God, whom he describes there as the Maximum. Out of the features of the Maximum, paying particular attention to his unity when he writes that "(...) for in the Maximum all difference is identity" (Nicholas of Cusa, 1985, p. 35). The unity is "(...)

exhibited by the infinite circle, which is eternal, without beginning and end, indivisibly the most one and the most encompassing" (Nicholas of Cusa, 1985, p. 35). The center of an infinite circle is the beginning of everything, the infinite circumference encapsulates everything, and the infinite diameter penetrates everything.

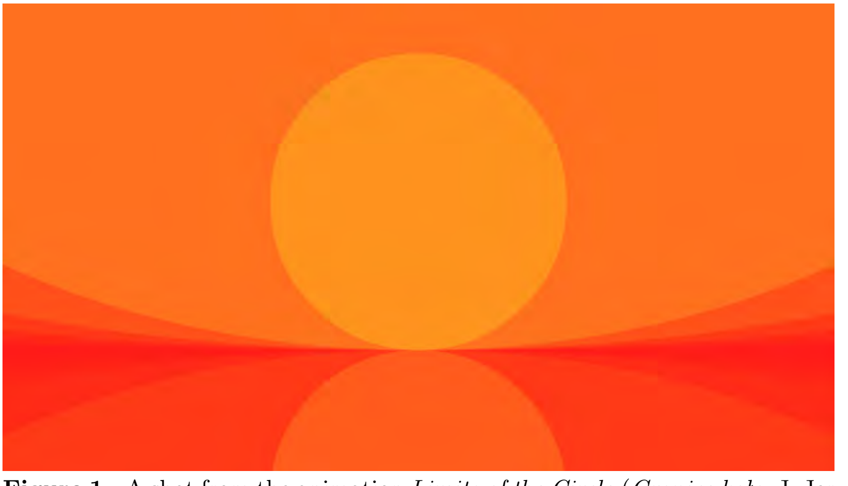

Figure 1. A shot from the animation *Limits of the Circle* (*Granice koła*, J. Jernajczyk, 2015).

Revolution of a circle around the diameter creates a sphere. It may be said that the sphere is potentially included in a circle, and using Nicolas of Cusa's terminology it is enfolded inside a circle. The finite circle is, of course, a sphere only in a potential sense – it has the ability to revolve. However an infinite circle is an actual sphere. That is why the spherical pertains here to the existence of the Maximum in the act. Just as the sphere is an actual line or a circle, in the same way the Maximum is supposed to be the actuality of all things (Nicholas of Cusa, 1985, p. 38).

The ontological metaphor of a sphere returns and is developed in one of the later works of Nicolas of Cusa – in the dialog *The Bowling-Game*. The description of a game popular in his times became a starting point for deliberations on the properties of the world and God.

Roundness leads to eternity. Both in a circle and in a sphere it is impossible to distinguish a point which may be the beginning or the end. Circles and spheres which have no beginning and no end become the model of eternity. Because the most perfect form of eternity in the Cusanus' understanding is God, the sphericity – like for Plato – is related to the divine. In the Nicolas of Cusa understanding the world is spherical, therefore it is eternal (Nicholas of Cusa, 2000, p. 1189–90). It is not however a perfect sphere; it is created on the basis of a model of a perfect sphere.

The roundness of a perfect sphere is not visible (Nicholas of Cusa, 2000, p. 1185):

> For since the surface of a [true] sphere is everywhere equally distant from its center, the outer-extremity of what is [perfectly] round-given that it ends at an indivisible point-remains altogether invisible to our eyes. For we see only what is divisible and quantitative.

We may not see a single point and what is visible may not consist of points. So if the limits of a perfect roundness are set out by an indivisible point, according to Cusanus, it may not be visible. Although roundness as such may not be seen, that does not mean that a round thing cannot be seen. Indeed, we see what is material, whereas in the material only the image of roundness is realized and not the true roundness (Nicholas of Cusa, 2000, p. 1186).

Roundness in itself includes the ability to move. Round objects move more easily than non-round, angular objects. In order to move a real sphere needs a mover, someone who will provide its impetus. Whereas perfect roundness does not need an external mover, it is able to move on its own, it is both the moved one and the mover. What is interesting, for Nicolas of Cusa the soul also moves (the movement of soul is life) in a circular motion. The movement of the soul returns to itself, as in the case of thinking about thinking, it moves itself. If the movement of a soul is circular and as we mentioned before if a circle is eternity, then the life of a soul is perpetual (Nicholas of Cusa, 2000, p. 1197). Cusanus proves at this point [6]. The immortality of the soul mainly on the basis of the roundness of its moves.

The metaphor of a sphere finds its reflection also in ethical deliberations. In the game described in the dialog, a bowling-ball has to be thrown from an agreed place so that it stops closest to the center of a previously drawn circle. Around the central point there are larger and larger circles drawn, which are given a specified number of points. Depending on the circle in which a bowling-ball stops, the player gets the respective amount of points (the closer to the center – the higher the score). The one who obtains 34 points first is the winner. The format of the game symbolizes the movement of our soul. Each bowling-ball sets out an adequate shape of movement, there is also no possibility that two bowling-balls

6. The notion of proof is not used here in a strictly mathematical sense; it rather stands for speculative reasoning in the scope of metaphysics or mathematical mysticism.

could stop at the same place. The aim of the game is the same as in life: to get as close as possible to the center of the circle which symbolizes God. In order to ensure that the bowling-ball stops near the center, the player must have some experience, which may be achieved only thanks to continual attempts. It is similar to one's exercising in virtue (Nicholas of Cusa, 2000, p. 1029):

> (...) each man, by exerting himself, must govern the inclinations and tendencies of his own bowling-ball. After a while, made temperate in this manner, he strives to find a way whereby the curvature of his bowling-ball does not prevent its arriving at the Circle of Life. This is the symbolic power of the game: that even a curved bowling-ball can be controlled by the practice of virtue, so that after many unstable deviations of movement, the ball stops in the Kingdom of Life.

In the dialog *The Bowling-Game* a famous metaphor also appears, the traits of which Borges was tracking in the essay referred to above "The Fearful Sphere of Pascal". In the version presented by Nicolas of Cusa instead of a sphere a circle is present: "(...) God is a Circle whose Center is everywhere (...)" (Nicholas of Cusa, 2000, p. 1226). The second part of the metaphor describing the circumference not existing anywhere was here omitted. It is worth observing that the metaphor makes sense both in the case of a sphere and a circle, because the center and the circumference are the basic properties of both objects. Only the dimension is different in which the two versions of metaphors are submerged. For those to whom the three-dimensional dimension seemed to be the highest one, a sphere (a closed ball) had to be the most appropriate object to present God. However today, assuming the existence of infinite dimensions, we would not tie the idea of perfection to a particular dimension, so we would not relate more perfection to a three-dimensional sphere (a closed ball) than to a two-dimensional circle (a disk). Perhaps also the thought of Nicolas of Cusa who did prove the identicality of a circle and a sphere, would have gone that direction[7].

7. Rudiments of many concepts characteristic for contemporary mathematics may be found in the papers of Nicolas of Cusa, among others from the area of non-Euclidean geometry and topology.

3. Circle in epistemological metaphors

3.1. Scientific knowledge as a dynamic circle

Willard Van Orman Quine, in *Two Dogmas of Empiricism* – one of the most influential philosophical essays of the 20th century – states that: "(...) total science is like a field of force whose boundary conditions are experience. A conflict with experience at the periphery occasions readjustments in the interior of the field" (Quine, 1951, p. 39). Explaining the interior structure of the area of the circle of scientific knowledge, in particular the relations between its outline and the interior, Quine writes: "The edge of the system must be kept squared with experience; the rest, with all its elaborate myths or fictions, has as its objective the simplicity of laws" (1951, p. 42). The *force field* referred to by Quine is naturally associated with a certain roundness – a circle or a sphere. Bearing that in mind Michał Heller developed the following metaphor: (1997, p. 7)

> If following Quine we compare scientific knowledge to the interior of a circle, what is yet unexamined will remain on the outside of the circle, and the circumference of the circle will be the boundary of knowledge – a place in which our knowledge meets ignorance. The circumference of that circle is constructed out of scientific questions – problems arising out of what we know (from the interior of the circle) but directed towards our ignorance (towards the exterior of the circle). Along with the advance of knowledge, in line with the growth of scientific achievements, the circle symbolizing scientific knowledge widens. Let us note however that at the same time the circumference of that circle grows too – the number of question marks increases! [8]

It should be emphasized that there is a significant difference between the two approaches. As long as Quine does not specify the shape and the size of the force field expressly focusing mainly on the changes that go on inside it (as a result of the reaction with experience), Heller accentuates the very process of the circle's circumference growth, which would indicate accepting the cumulative model of knowledge. Whereby, obviously, Heller's metaphor does not exclude the modification of the inside of the circle.

8. Transl. A. & J. Hamilton.

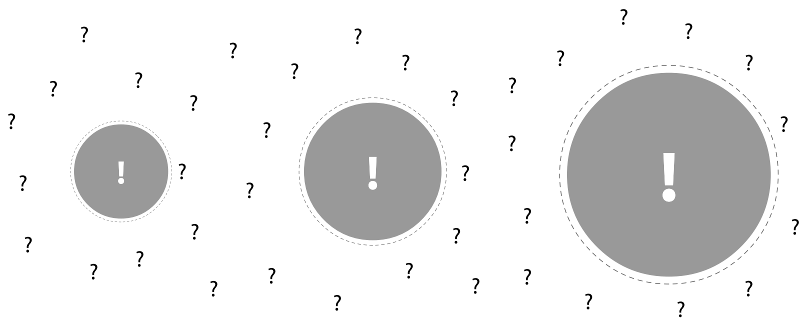

Figure 2. Model of evolution of a growing circle of knowledge according to Heller.

The increase of new questions noted by Heller which continuously accompanies the progress of scientific knowledge allows for a new – dynamic – reading of the Socratic maxim “I know that I know nothing”. If we recognize that along with the advance of knowledge the number of unsolved problems also grows and that the process may have no end, the constatation that the area of ignorance indefinitely surpasses the area of our knowledge is justified. The vision presented here seems pessimistic at first glance – we will never reach or at least approximate the completeness of knowledge about the world. Nevertheless, this metaphor may also be read as optimistic, as indicating the potential for perpetual development. If there are always new problems which require solving, the scholars will never rest in their quest. Such an optimistic overtone we can find in the words by Bertrand Russell, who described the consequences of the discovery of non-Euclidean geometries in the spirit of Heller’s metaphor (Russell, 1912, pp. 230–231):

> Thus, while our knowledge of what is has become less than it was formerly supposed to be, our knowledge of what may be is enormously increased. Instead of being shut in within narrow walls, of which every nook and cranny could be explored, we find ourselves in an open world of free possibilities, where much remains unknown because there is so much to know.

The metaphor of the circle of knowledge also constitutes an interesting illustration of relations which take place between the science and philosophy. According to Russell (1912, p. 240) “those questions which are already capable of definite answers are placed in the sciences, while those only to which, at present, no definite answer can be given, remain to form the residue which is called philosophy”. Referring to Heller’s metaphor we may say that the domain of philosophy spreads right beyond the circumference of the circle of knowledge.

3.2. Approximation of an ideal circle in metaphors of the pursuit of truth and knowledge

Nicolas of Cusa assumed a geometrical circle as the symbol of absolute truth, whereas the human quest to learn the truth he presented as a process of approximating the circle by polygons inscribed in it (Nicholas of Cusa, 1985, p. 8):

> Hence, the intellect, which is not truth, never comprehends truth so precisely that truth cannot be comprehended infinitely more precisely. For the intellect is to truth as [an inscribed] polygon is to [the inscribing] circle. The more angles the inscribed polygon has the more similar it is to the circle. However, even if the number of its angles is increased ad infinitum, the polygon never becomes equal [to the circle] unless it is resolved into an identity with the circle.

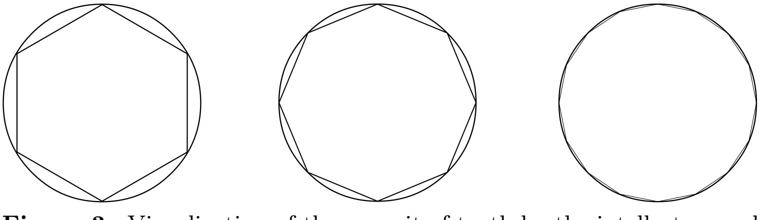

Figure 3. Visualization of the pursuit of truth by the intellect according to Nicolas of Cusa.

This metaphor is based on the mathematical *method of exhaustion* known since antiquity which has been used to approximate the area of figures and volume of solids of which it was not known how to measure them in a direct way. In order to approximate the area of a given figure, simpler figures of which the area it was known how to measure were circumscribed about and inscribed in that figure. The action was similar in terms of solids. Eudoxus is considered the inventor of the method of exhaustion whereas it was perfected by Archimedes who was able to approximate the area of a circle with astonishing accuracy by circumscribing about and inscribing regular polygons into it, starting with a hexagon and ending with as much as a 96-sided regular polygon (Katz, 2009, p. 101). Along with the increase of the number of angles, the values of the areas of the circumscribed and inscribed polygons approach each other and the area of a circle included within them was approximated more and more precisely (fig. 4).

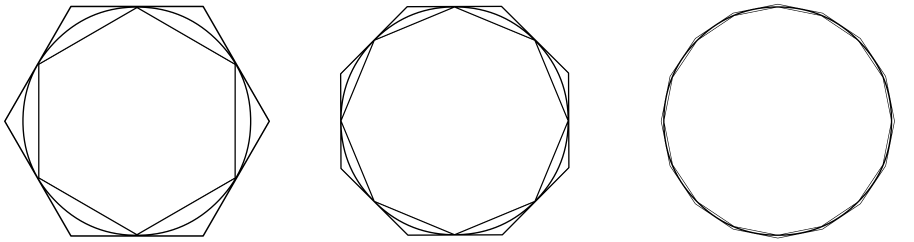

Figure 4. Area of a circle approximated in the method of exhaustion by circumscribing and inscribing regular polygons in it.

Although the method of exhaustion theoretically enabled the calculation of the circle's area with an arbitrary accuracy, its absolute value could not be measured[9]. It is not surprising then that Nicolas of Cusa employed the visualization of that method (fig. 3) in his metaphor of pursuing the truth. In illustrating the pursuit of human intellect he limited it only to polygons inscribed in a circle since the circle here symbolizes the truth about the world which is the property of God (or even is identified with God). It would be incorrect to illustrate human intellect in the form of circumscribed polygons because they transgress the limits of a circle.

Contemporarily it is assumed that all the geometrical properties of a circle have already been discovered (Davis and Hersh, 1981). What is interesting is that when we visualize that figure, in practice we use the archaic method of approximation which is much less subtle than the ancient method of exhaustion. On digital screen circles, ellipses or any curves are represented with the help of tiny squares – pixels. What to us seems like a smooth circle in reality is only its illusion. The visual effect of smoothness achieved through increasing the resolution of the matrix (increasing the number of pixels and decreasing their size) but also through using additional algorithms of optical edge smoothing which is called *antialiasing* (Hearn and Baker, 1997, pp. 171–180).

With reference to Heller's growing circle of knowledge and the discrete structure of screen matrix, an artistic work was created which is an individual visual metaphor of the evolution of scientific knowledge (*The Circle of Knowledge*/*Koło wiedzy*, J. Jernajczyk, 2008). The idea of complete knowledge was expressed here similarly to the way it was done by Cusanus, in the form of a circle. The limited human knowledge is presented by a figure built out of pixels which approximates an ideal circle.

9. Currently we are also unable to measure the value of a circle of a given radius, because in the ratio of the area and the radius the irrational number pi is included, which cannot be represented as terminating or repeating decimals. Every final result is then only an approximation.

The evolution of knowledge corresponds with the process of smoothing the edges of that figure. The first rough approximation of a circle is a square which through dichotomist divisions that take place in two dimensions slowly start transforming into a shape which resembles a circle more and more. As a result of consecutively dividing the pixels which are its constituents are becoming ever finer. The ones which are located outside the limits of the model circle are rejected and the entire figure is smoothed (fig. 5). Despite the fact that at some point we may have the impression that we are looking at a smooth circle, blowing up any fragment of the figure reveals that the pixels exist there permanently. The smoothing process, similar to the development of knowledge, potentially may go on *ad infinitum*[10].

Figure 5. Selected stages of a figure's evolution in the interactive installation *The Circle of Knowledge* (*Koło wiedzy*, J. Jernajczyk, 2008).

This metaphor similar to the metaphor of Cusanus is Platonic in character because it assumes that the ideal of absolute knowledge exists. Here it is presented as a circle which is the aim one should pursue. The essence of that evolution is not expanding the area of the figure like it was in Heller's metaphor but smoothing its edges. The development of knowledge is therefore presented here as a process of making continuous corrections and detailing and not a cumulative process which consists in the permanent growth of factual resources. An essential element of that process is rejecting former statements and convictions which along with the development of science have turned out to be outdated[11]. Falsified statements are symbolized here by falling off pixels which are beyond the perimeter of the ideal circle.

10. Three epistemological metaphors based on the circle which are discussed in this section, pertain only to the relationship between the "knowledge field" and experience. They describe what happens on the contact point of knowledge and ignorance. They do not discuss changes which happen inside that field. Quine pointed out in his essay that under the impact of experience, the internal organisation of the knowledge field is changed (Quine, 1951).

11. An example of such obsolescence of knowledge was the fate of Aristotelian physics.

Figure 6. *The Circle of Knowledge/Koło wiedzy* (J. Jernajczyk, digital print, 100 × 100 cm, 2008). Graphic version presenting 4 stages of the figure's evolution in one image (dynamic version can be find on: https://youtu.be/TIw5o2Gjaks).

3.3. Metaphor of the research circle by Kazimierz Twardowski

Considering the two key aspects of the circle – its center and circumference – for each direction set out by radii, two directions may be distinguished: from the center to the circumference and from the circumference to the center.

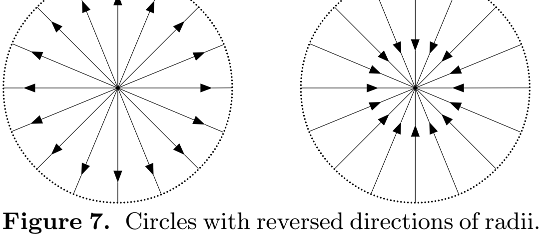

Figure 7. Circles with reversed directions of radii.

Heller's metaphor presented above – of the development of knowledge as a continuously growing circle – is an example of the first variant. Kazimierz Twardowski, the founder of the famous Polish philosophical school called the Lviv-Warsaw School, describing the process of cognition, focused on the direction from the circumference to the center. During the first meeting of the Polish Philosophical Society he is known to have said (Twardowski, 1904, p. 241):

> Like all radii of a circle, although they come from different points of the circumference, connect and meet in the center of the circle, the same is for us who want for all the directions of work and philosophical beliefs in our Society to pursue that one aim, the exhibition of truth [12].

Twardowski in that fragment refers to a circle as a circle of exercising philosophical knowledge, the circle of practicing philosophy. Despite the variety of specialization, diversity of approaches and ways of thinking, all of them should equally pursue true cognition. The truth, which is the circle's center, has therefore become the aim of philosophical pursuit.

The interpretation of Twardowski's visualization may be easily expanded if we consider not only the areas of philosophy and truth as a property of judgments. On the grounds of the high specialization of all detailed sciences and the atomization of scientific research arising out of that, research projects which combine more than one field of science have been created. Specialized sciences like biology or sociology, although they use different methodologies, often research the same subject, for example, the human being – only in different aspects; biology in an organic aspect, sociology in the aspect of social behaviors. They are located in different parts of the circle's circumference however they direct their attention towards its center – towards one and the same subject. That subject may be a tangible one, but it could also be an abstract problem. The unity of the subject enables cross-field research.

Conclusions

The circle and the sphere – mathematical objects – in visual-philosophical speculations take up an important spot. As early as in ancient cosmogonies the world appeared to the philosophers as rounded and similar to a sphere. Perfection based on the idea of an equal distance between the circumferential points from the center was presented with the use of a circle and a sphere. Each point of the circle to the same degree is its

12. Transl. A. & J. Hamilton.

beginning and its end, therefore essentially there is no beginning or ending point. As a result of that the circle has been and still is the symbol of eternity, everlasting without beginning or end. In popular geometrical approaches, God, the highest entity, was presented as an infinite circle or sphere, the center of which is everywhere and the circumference nowhere.

The circle may also serve as a visualization of cognitive processes. Presenting the development of scientific knowledge in the form of an ever-growing circle aids to visualize that along with the progress of knowledge – the growth of the circle's area, its circumference which sets out the scope of ignorance also grows. On the other hand, the regular polygon with a larger and larger number of sides and the figure constructed out of ever finer pixels, during further divisions become visually indistinguishable from the circle which defines their limits, however, they will never be identical with it. These metaphors reflect the essence of practicing knowledge in the classic sense, i.e. the essence of pursuing the truth – although we are closer to the truth, reaching its complete cognition will be impossible. The circle also helps to express the unity of cognition independent of how many people (or groups of scholars) participate in that process. The points of circumference symbolize various points of view, however from each of these points of view one and the same point may be seen: the circle's center. That point links various societies of researchers and that metaphor visualizes the interdisciplinary character of contemporary science. Amongst the points of circumference which are at the same distance, there is also room for an artistic perspective which complements the scientific image of the world in a creative way.

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Paola Vighi

Abstract paintings, objects, and actions: how promote geometrical understanding

Abstract. This paper presents the description and the analysis of an activity realised in kindergarten with 5-year-old children. The starting point is a painting of Wassily Kandinsky, titled "Soft Hard", and its reproduction made by the pupils, following particular tasks assigned by the teacher. Different pedagogical and psychological studies about visual perception deal with the laws of visual data organisation. They show that visual perception may hinder the ways of seeing figures; in other words, young pupils observe certain figures more than others in a picture. With reference to the perception of images of children who are $4-8$ years old, some pedagogical studies show that in the presence of abstract art works, the pupils show a "referential need" to identify a likeness, in order to find out what object there are in the painting. Starting from this assumption, in collaboration with a kindergarten teacher, I planned an activity based on a copy of each geometrical shape present in the chosen Kandinsky painting, and its reproduction, made by gluing the shapes onto a sheet of paper. From a mathematical point of view, this activity not only involves concepts such as "top or bottom" and "front or back," but also symmetry and geometrical transformations such as rotations and similarity, as well as the mutual positions of the geometrical shapes on the piece of paper. The analysis of the results provides us with interesting information about a child's approach to geometrical understanding.

Key words and phrases: concept of space, isometries, art, geometrical figures.
AMS (2000) Subject Classification: Primary 97D30, Secondary 97C80.

1. Introduction

In recent times, the diffusion of communication by the mass media has led researchers to study the role of different languages and systems of representation as new important aspects in education. “Art education” has been present in Italian primary school curricula since 1985. The main idea is that the artistic culture has a formative role, as documented by research on art and perception. In particular, in the National Indications for the curriculum of kindergarten schools (4 September 2012, p. 20), we can read:

> The encounter of children with art is an occasion for them to observe the world that is around them with different eyes. The materials explored by the senses, [...] the observation of works (paintings, museum, [...]) help to improve the perceptive capabilities, to cultivate the pleasure of the fruition [...] and to approach the culture and the artistic heritage.

In other words, while previously attention was focused on the ’production’ (spontaneous drawing, etc.), now the ’fruition’ is also important: looking at it not as a passive action, but as a dynamic activity of shapes, colours and, configurations selection.

The interactions between mathematical and artistic experience offer a broad field for exploration. In particular, the idea of this paper arises from the abstract art of Wassily Kandinsky (1866-1944). He affirms that art originates from space and time and he identifies the concepts of ’point’, ’line’, and ’surface’ as fundamental.

For Kandinsky, the issue of abstraction was propelled by the desire (or need) to explore the qualities of form, line, colour and facture as independent formal entities, rather than as pictorial elements for the depiction of subjects from life (Guy et al., 2007, p. 28).

The paintings of this artist are particularly suitable for a mathematical investigation, since they contain many traditional geometrical shapes placed in reciprocal positions following specific, from an artistic and aesthetic point of view, choices. The pleasant colours used and the apparent simplicity of the paintings can give very young students the motivation to observe and to investigate them. Nevertheless, following Arnheim (1987, p. 143), the images do not explain themselves, so it is necessary to study and plan activities which allow for observation and fruition. In other words, the first approach to an artistic painting can be promoted by the questions: “Do you agree with this painting? Why? What can you see? What does it represent?” During the second time, it would be suitable to propose an activity of ’reading the painting’ (fruition), possibly motivated by the need of its reproduction. The didactic activity described in the paper is based on this idea.

2. Theoretical framework

The first approach to geometrical concepts occurs in the so-called "physiologic space", as when the child sees and touches objects, it moves them. Following Van Hiele's theory (1986), the educational process passes through different levels. The first one is the "visual level", in which concepts develop starting from the observation of reality. It is very important for spatial knowledge, in which pupils recognise figures and are able to represent them as mental images. According to psychological studies, perception plays a fundamental role in the visualisation process:

> [...] by using perception, the visual thought organises itself as the starting point of insight and reflection, as well as mental activities which contribute to the formation of concepts (Marchini et al., 2009, pp. 62-63).

Perception is a process of selection and organization, of cognitive activities connected with knowledge and understanding. Nevertheless, visual perception may hinder the way of seeing geometrical figures. Following Duval (2005), this depends on the activity in which one is involved. In reference to the reading of images made by 4-8 year old children, Mazza (2001, p. 58) writes:

> The subject or the colour seem to be the parameters that determine the preference of pupils at this age. Even in the presence of abstract art works pupils show a "referential need" to identify a likeness, to find the object which "hides itself" behind the apparent oddity.

An important concept involved in painting, starting from its planning to its realization, is the 'concept of space.' The canvas is an empty space which must be organized by placing objects (independent space). In other geometrical situations, the figures create the space (not independent space): "[...] essentially, or primarily, we think about objects (or about shapes); space only coexists with them" (Speranza, 1997, p. 130). Usually, the space in a painting is a "microspace", namely a space that is manageable with hands and eyes (for instance, a sheet of paper). Sometimes it is also a "mesospace", manageable only with the eyes (i.e. a wall in a room).

In this space, the geometrical transformations are more important than the figures, since they allow for understanding. In particular, among the isometries, symmetry is a very complex topic. Research document the difficulties observed in its understanding. Piaget and Inhelder (1947) point out the individuation of a 'vertical axis' of symmetry in very young pupils. It could be a didactical obstacle, as Brousseau (1983) shows. Swoboda (2011) highlights the difference between the static and the dynamic approach to axial symmetry in pupils who are 4-6 years old. She made an experiment regarding the creation of a pattern using printed tiles. Firstly, the tiles were 'equal.' The second time, she placed a 'symmetrical tile' in the pattern. She observed that the children, when asked to reconstruct the regularity of the pattern, were trying to rotate the new tile instead of overturning it.

The research questions are:

– Which concept of space emerges from what the children produce?
– Do pupils use geometrical transformations when reproducing the painting? How?

3. Methodology

The presented research has been performed in a kindergarten school as the conclusion of a didactic itinerary about geometrical knowledge. It involved fourteen 5-6 year old pupils, who worked in groups or individually. The initial idea was to use paintings of abstract art to verify the acquisition of concepts by the pupils. In particular, the teacher [1] presents (Fig. 1) the following painting of Kandinsky (Fig. 2) entitled “Soft Hard” (1927).

Figure 1. Teacher shows the painting.

Figure 2. “Soft Hard” by Kandinsky (1927).

1. I wish to thank the teacher Palma Rosa Micheli, Scuola dell'Infanzia Statale “Lodesana”, Fidenza (PR), Italy, for her collaboration and helpfulness.

The artist explains it in terms of the contrast between the softness of the blue background and the hardness of geometrical figures.

In the classroom, the teacher asks the pupils to observe the painting and to reproduce it, using paper, scissors, and glue. Didactically, the work involves the recognition and use of geometrical shapes (squares, rectangles, triangles, circles, and shapes with curved boundaries). The geometrical figures usually presented in kindergarten are those suggested from the structured materials for learning mathematics named “Logic Blocks of Dienes”. Often, pupils use them to realise free composition, castles, cars, and so on. In this work, the composition is obliged; it is necessary to reproduce the Kandinsky painting. It is a very difficult activity that imposes the observation of the shapes, their reciprocal positions, their possible superimpositions, their arrangement on the sheet of paper that represents the background, and possibly using geometrical transformations (rotations, translations, symmetries).

There are about 20 geometrical shapes in the painting: 1 square, 2 rectangles (the third rectangle can be seen on the right, and it is obtained by connecting two triangles), 2-3 circles, 12 triangles, 3 “moons” (this word was suggested by the children). The square is placed in a particular way: two sides are parallel to the hypotenuse of a ’big’ red triangle. It is the only figure (except the circles) with its sides not parallel to the sides of the rectangular background. On the contrary, all of the rectangles are drawn with their sides ’horizontal’ and ’vertical’ regarding the background. There are three equilateral triangles, four isosceles triangles, and six right-angled triangles. The three “moons” are typical to Kandinsky drawings.

Surely, there is an idea of ’non-isotropic space:’ horizontal and vertical directions permeate the painting. It is also a ’space limited’ by the border of the sheet of paper.

We can also observe the presence of three symmetrical compositions of shapes. On the left, the axis of symmetry is a straight line containing the axes of the triangles. On the right, there is a ’moon’ superimposed on an isosceles triangle: its axis is the axis of symmetry of the composition. The three triangles arranged as a tree are symmetrical.

The pupils must observe the painting, look for a shape, and manipulate it with the aim of getting the ’right orientation’ on the sheet of paper as well as the correct disposition with respect to the other shapes. This way, they make use of topological concepts such as “inside or outside” and the concepts of “top or down” or “forward or back”, “on the left or on the right” and so on. They must use geometrical transformations (translations, rotations, symmetries) and apply them to the figures.

The children have two possibilities of working: gluing the pieces one by one (Fig. 3), or organizing the disposition of the shapes on the background and gluing them at the end (Fig. 4). Obviously, the second way allows for better organization and a better result.

Figure 3. Reproduction without planning.

Figure 4. Reproduction with planning.

4. Results

Here, I analyse only the works of three pupils, in terms of the following criteria:

1. Orientation of shapes with respect to the sheet of paper representing the background of the painting.
2. Use of geometrical transformations.
3. Arrangement of triangles in copy.

Gabriele's work and its analysis (see Fig. 5).
Globally, the reproduction respects the original disposition of shapes: if the age of the boy is taken into account, we can say that it is a good copy. Criterion 1: the orientation of the shapes is fundamentally respected, with the only exception being the bicoloured rectangle (which is also separate from the bigger triangle). Most of the sides are not 'horizontal' or 'vertical' with respect to the background, but it could be caused by a problem with manipulation during the operation of gluing. Criterion 2: the symmetries are respected. The square is rotated, though casually. The

Figure 5. Gabriele's work.

smallest triangles situated on the right and on the bottom of the biggest triangle are overturned and their colours are swapped. It could be a sign of a problem connected with the use of the symmetry which is an inverse isometry. Criterion 3: the sequence of triangles in the tree is incorrect, since the equilateral should be on the top.

Gaia's work and its analysis (Fig. 6). She recognised a 'sailing boat' on the right of the Kandinsky painting and, therefore, in her copy, she tried to make a boat, assembling some pieces nearby. Criterion 1: on the right, the horizontal and vertical directions are respected; not in the other parts. Criterion 2: on the left, the big triangles are without a common axis of symmetry. The square is crosswise. On the right part aloft, the half-moons are placed in an incorrect position in comparison to the biggest triangle, they are also swapped. Criterion 3: the sequence of black triangles in the 'tree' is incorrect; furthermore, there are two equilateral triangles instead of one, and the bigger one is in the middle.

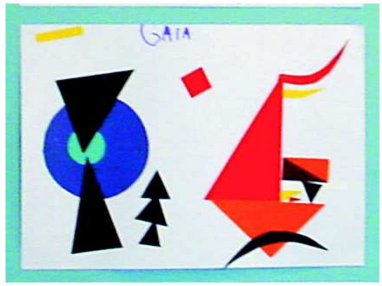

Figure 6. Gaia's work.

Dylan's work and its analysis (Fig. 7). It is an interesting and original work. Criterion 1: the square appears to be in a suitable orientation and position. Also, the rectangles are well-oriented. Criterion 2: The two-coloured rectangle is overturned and 'rotated.' It could be a problem connected with symmetry. The smallest triangles are 'symmetric.' Criterion 3: the management of the triangles shows that this child distinguishes neither their shapes (equilateral or isosceles) nor the sizes (small, medium, big). Consequently, the triangles placed over the circles stay in the interior of the bigger circle and the tree appears to be very different from the original. It seems that the word 'triangle' (and maybe the black colour) guide the production of this boy, who seems to use a black triangle without observing its features. In fact, in kindergarten, when a teacher presents the triangle, typically it is an equilateral one.

Figure 7. Dylan's work.

The analysis of the protocols seems to show that the sides of the background do not constitute a reference point for these young pupils. It is well known that for young children the space is 'not independent:' firstly, there are objects; the space is 'created' by these objects. Therefore, the sides of the sheet of paper are not used to obtain 'horizontal' or 'vertical' directions. On the contrary, in primary schools, their influence is very strong: is it a problem related to the concept of space and/or of the didactical contract?

The failure to respect the axes of symmetry in the children's productions could be explained by the need to put the shapes in the position 'object-to-object', one next to another, which leads to ignoring the symmetry. Obviously, it is also a problem of the manipulation of the shape, the stick of glue, and the sheet of paper that represents the background.

5. Conclusions

According to Vygotsky (1987), we live among things, facts, and phenomena and we confer a meaning to them, depending on our mental and cultural schemes. From a didactical point of view, the challenge is to stimulate curiosity with the aim of enriching our mental schemes with new knowledge and meanings. In particular, this work documents how looking becomes investigation and research if it is supported by objects that stimulate in this sense. Abstract paintings, with their features of shapes, colours, and composition, provoke this 'transgressive way' of seeing the world through mathematical eyes.

The experiment documents that young pupils are able to observe and explore the painting projected in the mesospace of the wall, and they are capable of reproducing it on the microspace of the sheet of paper, arranging the shapes in a "suitable way." Certainly, the colours and the dimensions (small, medium, or big) of the figures help them in the choice of the pieces used in the work. The arrangement of the figures pre-exists in the space and creates it. Therefore, the young pupils conceive the space as "not independent".

Work with unusual shapes appears difficult. In kindergarten, the figures manipulated above all are circles, squares, rectangles, and equilateral triangles. The equilateral triangle, having three axes of symmetry, can be manipulated in a better way than an isosceles (with only one axis of symmetry) or a right-angled triangle (without any symmetry): the equilateral one can be placed on the sheet of paper without any particular attention, and to arrange an isosceles, rotating it is enough, while sometimes the right-angled triangle must be also overturned if we want to place it properly. In other words, the difference between the use of direct or inverse isometries emerges from these protocols. Didactically, inverse transformations appear to be more difficult to apply in a situation of manipulation. The research on this topic documents this aspect. This experience confirms that the idea of overturning the shapes is nearly absent in very young pupils (Swoboda, 2011).

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